A TREATISE ON

SURVEYING

COMPRISING THE THEORY AND THE PRACTICE

BY

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PART II

HIGHER SURVEYING

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PREFACE.

THIS treatise on "Higher Surveying" forms Part II of Gillespie's Surveying. "Land Surveying and Direct Leveling" form Part I.

In the preparation of this volume the writer is indebted for important contributions to several friends who are specialists in the subjects they have here discussed. The chapters on Geodesy, Field Astronomy, Trigonometric Leveling, and Precise Leveling were prepared by Assistant O. B. French, of the United States Coast and Geodetic Survey, and the methods of work given are those approved by the Geodetic Conference held in Washington in 1894. The chapter on Topography was prepared by Prof. F. H. Neff, of Case School of Applied Science; the chapter on Mining Surveying, by Mr. E. P. Dickey, Mining Engineer, Pittston, Pa.; and the chapter on City Surveying, by Mr. Horace Andrews, City Engineer of Albany, N. Y.

The methods recommended for the adjustment of observations are those given by Prof. T. W. Wright, in his work on "The Adjustments of Observations."

In connection with all subjects here treated will be found quite full references to a more extended treatment of special topics, which could not be fully discussed within the limits of this volume, the purpose being to aid students in their researches along special lines.

CADY STALEY.

CASE SCHOOL OF APPLIED SCIENCE,
CLEVELAND, OHIO, *October, 1897.*

GENERAL DIVISION OF THE SUBJECT.

[*A full Analytical Table of Contents is given at the end of the volume.*]

PART II.

HIGHER SURVEYING.

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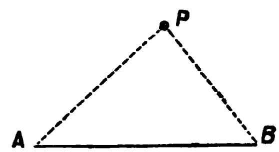
CHAPTER X.

TRIANGULAR SURVEYING.

PLANE SURFACES.

571. TRIANGULAR SURVEYING is founded on the method of determining the position of a point by the intersection of two known lines. Thus, the point P is determined by knowing the length of the line A B, and the angles P B A and P A B, which the lines P A and P B make with A B. By an extension of the principle, a field, a farm, or a country, can be surveyed by measuring only one line, and calculating all the other desired distances, which are made sides of a connected series of imaginary *triangles*, whose angles are carefully measured. The district surveyed is covered with a sort of network of such triangles, whence the name given to this kind of surveying. It is more commonly called "Trigonometric Surveying," and sometimes "Geodesic Surveying," but improperly, since it does not necessarily take into account the curvature of the earth, though always adopted in the great surveys in which that is considered.

FIG. 428.



572. Outline of Operations. A *base-line*, as long as possible (five or ten miles in surveys of countries), is measured with extreme accuracy.

From its extremities, angles are taken to the most distant objects visible, such as steeples, signals on mountain-tops, etc.

The distances to these and between these are then calculated by the rules of trigonometry.

The instrument is then placed at each of these new stations, and angles are taken from them to still more distant stations, the calculated lines being used as new base-lines.

This process is repeated and extended till the whole district is embraced by these "primary triangles" of as large sides as possible.

One side of the last triangle is so located that its length can be obtained by measurement as well as by calculation, and the agreement of the two proves the accuracy of the whole work.

Within these primary triangles, *secondary* or smaller triangles are formed, to fix the position of the minor local details, and to serve as starting-points for common surveys with chain and compass, etc. Tertiary triangles may also be required.

The larger triangles are first formed, and the smaller ones based on them, in accordance with the important principle in all surveying operations, always to work from the whole to the parts, and from greater to less.

When the survey is not very extensive, and extreme accuracy is not required, the ordinary methods of measuring lines and angles may be employed. For precise methods of measuring lines and angles see Chapter XI, GEODESY.

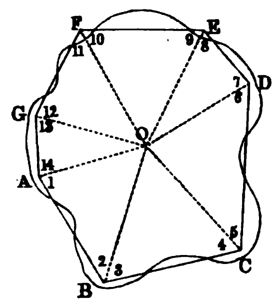
573. Calculation and Platting. The lengths of the sides of the triangles should be calculated with extreme accuracy, in two ways if possible, and by at least two persons. Plane trigonometry may be used for even large surveys; for, though these sides are really arcs and not straight lines, the difference will be only one twentieth of a foot in a distance of 11½ miles; half a foot in 23 miles; a foot in 34½ miles, etc.

The platting is most correctly done by constructing the triangles by means of the calculated lengths of their sides. If the measured angles are platted, the best method is that of chords. If many triangles are successively based on one another, they will be platted most accurately by referring all their sides to some one meridian line by means of "Rectangular Co-ordinates." In the survey of a

country, this meridian would be the true north and south line passing through some well-determined point.

574. Radiating Triangulation. This name may be given to a method shown in the figure. Choose a conspicuous point, O, nearly in the center of the field or farm to be surveyed. Find other points, A, B, C, D, etc., such that the signal at O can be seen from all of them, and that the triangles A B O, B C O, etc., shall be as nearly equilateral as possible. Measure one side, A B for example. At A measure the angles O A B and O A G; at B measure the angles O B A and O B C; and so on, around the polygon. The correctness of these measurements may be tested by the sum of the angles. It may also be tested by the trigonometrical principle that the product of the sines of every alternate angle, or the odd numbers in the figure, should equal the product of the sines of the remaining angles, the even numbers in the figure.

FIG. 429.



The triangles A O B, B O C, C O D, etc., give the following proportions [Trigonometry, Art. 12, Theorem I] : A O : O B :: sin. (2) : sin. (1) ; O B : O C :: sin. (4) : sin. (3); O C : O D :: sin. (6) : sin. 5; and so on around the polygon. Multiplying together the corresponding terms of all the proportions, the sides will all be canceled, and there will result

1 : 1 :: sin. (2) × sin. (4) × sin. (6) × sin. (8) × sin. (10) × sin. (12) × sin. (14) : sin. (1) × sin. (3) × sin. (5) × sin. (7) × sin. (9) × sin. (11) × sin. (13).

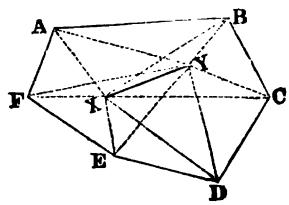
Hence the equality of the last two terms of the proportion.

The calculations of the unknown sides are readily made. In the triangle A B O, one side and all the angles are given to find A O and B O. In the triangle B C O, B O and all the angles are given to find B C and C O; and so with the rest.

Another proof of the accuracy of the work will be given by the calculation of the length of the side A O in the last triangle, agreeing with its length as obtained in the first triangle.

575. Farm Triangulation. A farm or field may be surveyed by the previous methods, but the following plan will often be more convenient: Choose a base, as X Y, within the field, and from its ends measure the angles between it and the direction of each corner of the field, if the theodolite or transit be used, or take the bearing of each, if the compass be used. Consider first the triangles which have X Y for a base, and the corners of the field, A, B, C, etc., for vertices. In each of them one side and the angles will be known to find the other sides, X A, X B, etc. Then consider the field as made up of triangles which have their vertices at X. In each of them two sides and the included angle will be given to find its content. If Y be then taken for the common vertex, a test of the former work will be obtained.

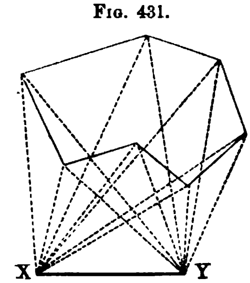
Fig. 430.



The operation will be somewhat simplified by taking for the base-line a diagonal of the field, or one of its sides.

575¹. Inaccessible Areas. A field or farm may be surveyed without entering it. Choose a base-line X Y, from which all the corners of the field can be seen. Take their bearings, or the angles between the base-line and their directions. The distances from X and Y to each of them can be calculated as in the last article. The figure will then show in what manner the content of the field is the difference between the contents of the triangles, having X (or Y) for a vertex, which lie outside of it, and those which lie partly within the field and partly outside of it. Their contents can be calculated as in the last article, and their difference will be the desired content. If the figure be regarded as generated by the revolution of a line one end of which is at X, while its other end passes along the boundaries of the field, shortening and lengthening accordingly, and if those triangles generated by its movement in one direction be called *plus* and those

Fig. 431.



generated by the contrary movement be called *minus*, their algebraic sum will be the content.

576. In all the operations which have been explained, the position of a point has been determined, as in Art. 6, by taking the angles, or bearings, of two lines passing from the two ends of a base-line to the unknown point. But the same determination may be effected inversely, by taking from the point the bearings, by compass, of the two ends of the base-line, or of any two known points. The unknown point will then be fixed by platting from the two known points the *opposite* bearings, for it will be at the intersection of the lines thus determined.

577. Defects of the Method of Intersection. The determination of a point by the method founded on the intersection of lines, has the serious defect that the point sighted to will be very indefinitely determined if the lines which fix it meet at a very acute or a very obtuse angle, which the relative positions of the points observed from and to often render unavoidable. Intersections at right angles should therefore be sought for, so far as other considerations will permit.

CHAPTER XI.

GEODESY.

578. Historical Sketch. The word geodesy signifies the art of measuring the earth. It was formerly applied to land surveying in general, but is now limited to surveys of such magnitude that the curvature of the earth can not be neglected, and to surveys undertaken for the purpose of determining the figure of the earth—i. e., its form and size.

Some of the Greek mathematicians believed the earth to be spherical, and attempted to determine its size by measuring the length of its circumference.

In the year 276 B. C. Eratosthenes made the first step toward the determination of the circumference. He observed that an object at Syene, in southern Egypt, cast no shadow on the day of the summer solstice, while at Alexandria the shadow cast showed that the sun made an angle with the vertical equal to one fiftieth of a circumference. Hence he reasoned that if the two places are in the same meridian the entire circumference of the earth must be fifty times the distance between the two places.

Other attempts of a similar character were made from time to time until, in 1615, Willebrord Snellius, in Holland, introduced the method of triangulation for determining the length of an arc on the earth's surface. He measured a small base-line near Leyden, and determined the length of an arc of the meridian by means of thirty-three triangles. As logarithmic tables were unknown at this time, it is easily conceivable that the work of making the computations and reductions was enormous.

Numerous arcs have been measured, particularly in the last century, and at the present time nearly all civilized nations are con-

ducting geodetic surveys of a high order. Europe is nearly covered with triangulation, as also sections of Asia, Africa, and America.

Nearly all the geodetic work in the United States has been done by the United States Coast and Geodetic Survey, which was first organized in 1807, but was interrupted by the War of 1812. It was reorganized in 1832, and has been carried on continuously since that time. Besides making such surveys as are necessary for the cartographic work along the coast of the United States, the Coast and Geodetic Survey has about completed the measurement of an arc of a parallel extending from the Atlantic Ocean to the Pacific, and also an oblique arc extending from Maine to the Gulf of Mexico, besides several shorter meridional arcs.

Two short arcs (a parallel and a meridian) have been measured in the region of the Great Lakes by the corps of engineers, United States Army.

579. General Principles. Geodesy may briefly be defined as that one of the applied sciences which has for its object the determination of the form and size of the earth, and of determining the geographical positions of points on its surface, usually referred to three co-ordinates, viz., latitude, longitude, and altitude, and which is occupied with problems or relations between such points. It is thus different from ordinary surveying, which excludes astronomical observations, and, within circumscribed limits, regards the geometrical surface of the earth as a plane.

Whatever the design of a geodetic operation, whether it be to define a portion or the entire surface of a country, or only its coast or boundaries, or whether its purpose be to measure arcs as a contribution to the data for ascertaining the figure of the earth, it must be based in all cases upon triangulation, the greater or less complexity of which will depend chiefly and necessarily on the hypsometric and physical features of the country. Triangulation is introduced because of the impracticability of measuring long distances over all kinds of country, with the required degree of accuracy and economy, by the known ordinary means for direct linear measurement.

Even by the method of triangulation we must measure at least

one line of the net or chain of triangles, as well as nearly all the angles, but we can usually select some site favorable for accurate linear measure and of such length that we will obtain the required degree of accuracy in the determination of the length of the sides of the triangles forming the principal net or chain. Having measured one side of the net of triangulation and at least two angles in each of the triangles, we may compute the remaining parts by trigonometric formulas. Having determined the length of any side, we may use this computed value as a base for the computation of any adjacent triangles into which this line may enter, and so on through the whole net.

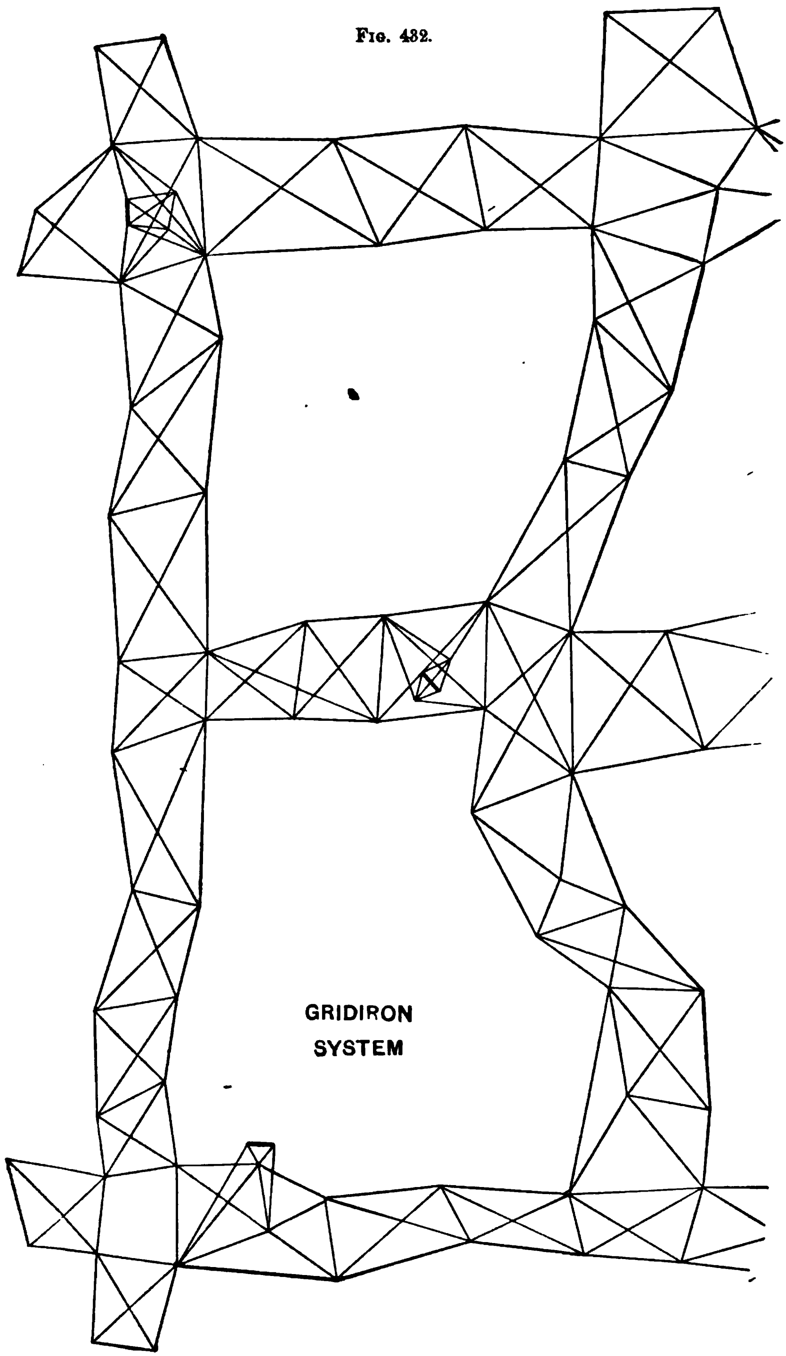
Triangles serve merely to determine the *relative* positions of points. In order to orient the net and obtain the *absolute* positions, we must determine the positions of one or more of the points by observing its astronomical latitude and longitude and also the direction of one or more of the lines forming the triangles. The third co-ordinate, the altitude, is obtained by leveling, as explained in the chapters on that subject.

Having obtained the absolute position of one point of a net of triangles (i. e., its latitude and longitude with reference to some fixed meridian), as also the azimuth of one of the lines radiating from this point (i. e., the angle it makes with the meridian of this fixed point), we may compute the absolute positions of all the other points of the net by means of Puissant's modified formulas. (See Position Computation, page 114.) If, however, the lines are more than about one hundred and ten kilometres (say seventy miles) in length, more rigorous formulas must be used if accurate results are desired.

All the work connected with the determination of absolute positions is known as triangulation. Hence, as mentioned above, triangulation is the basis of all geodetic work. It furnishes the relative and absolute geographical positions of points spread over extended areas for topographic or hydrographic surveys, location of boundaries, measurement of arcs for the determination of the figure of the earth, and for engineering operations in general.

580. General Systems of Triangulation. Two general systems of triangulation have been used whenever extended areas are to be

Fig. 432.

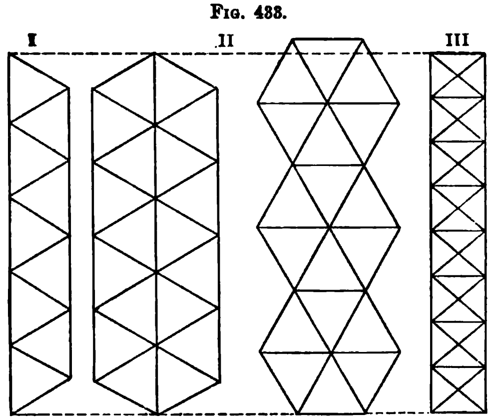


surveyed—viz., the so-called gridiron system and the central system. The former consists of a set of parallel chains of triangles crossed by another set at right angles to them, as shown in Fig. 432. The central system begins at the center of the area to be surveyed and extends radially in all directions, increasing by concentric rings and eventually covering the whole area.

The gridiron system is usually far preferable, as the work of computation and adjustment is more simple, and it admits of preference being given to the advance of work in the particular locality where it may be needed most. The gridiron system is used almost exclusively, both in this country and in foreign countries. The rectangular spaces of the gridiron system may be filled by subordinate triangulation whenever desired, thus covering the whole area as with the central system.

581. Triangulation Figures. Although the triangle is the fundamental figure of all triangulation, it is usually found advantageous to combine or intertwine them into complex figures in order to increase the accuracy, or area covered, or to subserve better the purpose for which the work is intended.

FIG. 433.



There are numerous forms of these complex figures, but only three are usually employed. The most simple form possible for connecting two points, as the terminals of an arc, is a chain of single triangles, as shown by I in Fig. 433. If the triangles are nearly equilateral, very good results may be obtained. Single triangles are objectionable, however, as affording no check upon the work beyond the condition that the

sum of the three measured angles of each triangle, less the spherical excess, must equal two right angles.

If the greatest attainable accuracy is desired, Form III must be used. It consists of a chain of quadrilaterals with the diagonals observed as well as the sides.

Form II will be found advantageous when areas are to be covered; the first arrangement when areas alone are to be covered, and the second arrangement when it is desirable to cover both areas and distances.

If chains of the three forms of figures are constructed, using the same maximum length of line—i. e., consider that a certain length of line is the maximum length that can enter, owing to the nature of the ground, efficiency of the instrument, and means available—then the relative values of the three forms of figures may be estimated by comparing their results for a given linear extent. Since nine triangles with a unit length of side reach nearly as far as three hinged hexagons with unit length of side ($3\sqrt{3} = 5{\cdot}20$), and slightly surpass seven quadrilaterals having diagonals of a unit length ($7\sqrt{\frac{1}{2}} = 4{\cdot}95$), we may consider that each covers practically the same linear distance, and form the following table of comparisons:*

		Linear distance.	No. of stations.	Total length of sides.	Area.	Conditions.
I.	Triangles, equilateral..	5·00	11	19·0	4·5	$(n-2) = 9$
II.	Hexagons, hinged.....	5·20	17	34·0	9·0	$\left(\frac{7n-14}{5}\right) = 21$
III.	Quadrilaterals, squares	4·95	16	29·6	3·5	$(2n-4) = 28$

This table shows that a chain of single triangles is the most simple and economical for covering distances only, requiring the smallest number of stations; that a chain or net of hexagons is best for covering areas, especially as the number of conditions is more than doubled, thus increasing the accuracy.

These "conditions" are rigid geometrical relations of the figures

* Taken from Mr. Schott's paper in "United States Coast and Geodetic Survey Report" for 1876.

themselves; hence, as the principal source of error in triangulation is in the determination of the angles necessary to fill these geometrical relations, the more of these relations we introduce the greater is the probability of obtaining the true figure.

As a chain of quadrilaterals (Form III) has the greatest number of conditions, it is used wherever great accuracy is desired.

All these forms of figures are ideally perfect for the purpose designed, and will rarely be found practicable in the field work. They will serve, however, as a standard that the geodesist should strive to reproduce as nearly as practicable.

When laying out a triangulation scheme in the field, it will often be found impracticable to maintain the uniform size of the scheme without the introduction of additional stations or construction of high signals, especially where quadrilaterals are desired. In nearly all such cases a quadrilateral, or pentagon, with an interior station from which all the others are visible, may be inserted, as shown in several places in Fig. 432. This makes a very strong figure, being unquestionably preferable to a contraction of the scheme or the introduction of poorly conditioned triangles.

582. Classification. Great diversity exists in the character of triangulation, depending upon the special object of the work, as, for instance, whether designed as a contribution to the measure of the figure of the earth or simply to give the relative positions necessary for the traverse work of a topographical survey. The first is purely geodetic in character, while the second is but little different from plane surveying. For the various classes of work different degrees of accuracy are needed, thus requiring different instruments, different methods of observation and computation, and the like. Hence the following more or less well defined subdivisions have been made, viz., Main and Subordinate Triangulation; or Primary, Secondary, and Tertiary Triangulation.

By the term Main Triangulation we mean the principal series of figures, composing a chain or net of triangles or polygons, which are necessary for the connection of the various points of the surface of the country under consideration.

By Subordinate triangles we mean those triangles which de-

pend upon the main series for their support, and which may be contained within it, or branch off from it in any direction.

The other classification is more technical.

583. Primary Triangulation. As the name implies, Primary Triangulation is the first in all respects. It is characterized by the greatest length of sides practicable in the country traversed and by the greatest accuracy of measure. The geodetic positions will depend upon the direct observation of astronomical latitudes, longitudes, and azimuths. All linear measures, as base-lines, must be made with great refinement. The length of triangle side must necessarily vary with the character of the country traversed, averaging in flat or slightly rolling country 30 or 40 kilometres (say 20 or 25 miles), and increasing in length as the country becomes more mountainous. In mountains, such as in the western part of the United States, lines averaging 150 kilometres (90 miles) may be obtained, and occasionally more than double this length. In the transcontinental triangulation, by the Coast and Geodetic Survey, a line 294 kilometres (183 miles) in length was used in the main scheme with excellent results, and another 309 kilometres (192 miles) in length was observed but not included in the main scheme. Peaks even more distant than this have been seen but are not practicable for triangulation, as they are only visible when the atmosphere is unusually clear.

The computation of triangulation developed on such a large scale must be very refined, and is consequently very laborious. It is necessary to consider the spheroidal shape of the earth, to reduce the observed horizontal directions, or angles, to what they would have been if made at sea level (due to the fact that the normals of the earth's surface do not intersect); to correct the astronomical latitude for elevation above the sea (the vertical being a curved line), etc. It is also necessary to employ formulas for computation of geographical positions of greater accuracy than Puissant's modified formulas; to use logarithmic tables more extended than seven places, etc.

584. Tertiary Triangulation. This is the lowest order of triangulation, and is designed to furnish the positions of points for

use in topographic or hydrographic surveys, or engineering purposes in general. It is usually of such limited extent that the earth's curvature has less effect than the errors of observation, and may therefore be neglected. Its sides are usually less than 15 or 20 kilometres (about 12 miles) in length.

585. Secondary Triangulation. Secondary triangulation is intermediate between Primary and Tertiary triangulation, serving to connect the long sides of the former with the short sides of the latter. The length of its sides may therefore vary considerably, depending upon the character of the work it connects.

It may sometimes happen that we wish to connect two points, and do not wish to make so elaborate a triangulation as the primary, and yet want a more accurate scheme than the tertiary; hence we use the methods employed in secondary triangulation, or, in other words, after defining the relation of the three classes to each other, and fixing the instruments and methods that may be employed in each class, we may designate any isolated piece of work according to the class it most nearly fits. It is erroneous to designate very large triangulation as primary unless the work is refined and will bear elaborate reductions; nor should it be considered as secondary or tertiary unless it is done with as great refinement as belongs to those classes. Hence triangulation, where only one or two readings of the angles are made with small instruments, belongs more properly to reconnaissance than to triangulation of any class.

586. Base-line Sites. The number of base-lines to be introduced into a triangulation, as well as the length of the bases, depends on the average length of the sides of the triangulation and upon the degree of accuracy desired for the latter. Since base-lines can be measured with much greater accuracy than can be sustained through a triangulation, they should be introduced as often as practicable when the best class of work is desired.

The more accurate the angular measures in a chain of triangles the greater may be the distance apart of its base-lines. This distance varies accordingly between wide limits, but ordinarily may be from twenty to forty times the combined length of the bases. Or,

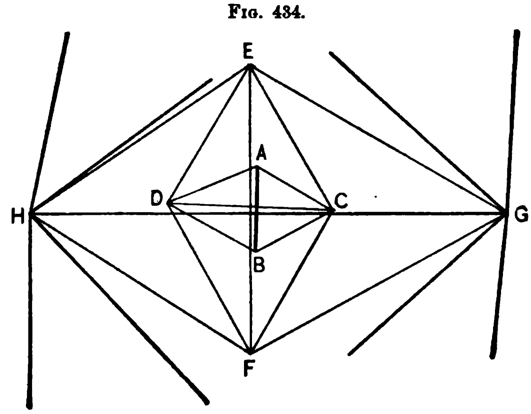
if the country will admit, base-lines may be introduced in about every eighth or tenth figure for primary work.

When secondary or tertiary triangulation is separated from the primary work, base-lines may be introduced at the beginning and end; or when the work is small and merely local, a single base may suffice.

No precise rule for deciding *a priori* on the length of a base-line can be given. Any of the fractions one tenth to one sixth of the length of an average side of the triangulation may be useful for an estimate, since the actual length of base-lines varies between wide limits. For a triangulation of the first order the length may be 15 kilometres (9·3 statute miles), or even more, and for a third-order triangulation it may be 1 kilometre (0·6 mile), or occasionally less. Base-lines between 5 and 8 kilometres (say 4 to 5 statute miles) in length are the most common for primary triangulation.

587. Connection of Base-line with the Triangulation. The transfer of the comparatively short length of a base to the greater length of a side of the triangulation is generally effected by several steps, so as to avoid too acute angles, and, consequently, loss of accuracy. The figure of the base net connecting the base with the triangulation is therefore one of importance, and ideally may be described as a series of quadrilaterals with diagonals intersecting at right angles, the length of these diagonals increasing, ordinarily, in a ratio of 1 to 2 or 3, thus requiring two or three steps to ascend to the length of a main line. Fig. 434 represents the ideal connection, A B being the base and H G the side of the main

Fig. 434.



triangulation. Additional conditions may be introduced by observing the lines A E, B F, C G, and D H, if they are open and the stations intervisible, thus increasing the accuracy with but little extra work, excepting in the computation.

588. Reconnaissance. Reconnaissance, as usually understood, embraces all those investigations of the region to be triangulated which precede the actual field work of measurement, and comprises the selection of the most feasible chain or net of geometrical figures, location of base-lines, determination of the appliances necessary, and the collection of all data that may be valuable in the prosecution of the work of the triangulation.

It should be thorough and exhaustive, developing all possible schemes, and should comprise all information affecting the economy and facility of the operations to follow. The officer in charge of the work should be a man of considerable experience in the class of triangulation he is to develop. Before taking the field he should make a careful study of all available maps of the region, noting the character of the country, lines of travel, location and relative importance of towns and villages, and particularly the drainage of the region, as this usually determines the character of the system of triangulation that may be employed.

Experience has shown that whenever a system crosses the drainage of a flat or rolling country it can usually be done only by contracting the scheme or elevating the stations by means of scaffolds or high structures; hence, whenever practicable, the system should always follow the drainage of a country rather than cross it.

589. Degree of Adaptability of Figures. The degree of adaptability of the several geometrical figures of triangulation to the orography of a region is quite different. Chains of single triangles easily adapt themselves to the most complex topography, whereas quadrilaterals with observable diagonals possess this quality in the least degree, and will often be found impracticable figures on that account. Pentagons or quadrilaterals with central points also conform readily to the configuration of a country, however complex it may seem. Polygons of more sides than five are usually not so advantageous, owing to the disposition of the stations, and tend to

retard the work. They are rarely used excepting on surveys where areas are to be covered rather than distances.

With the present methods and instruments the tendency seems to be toward the introduction of simple well-conditioned triangles in a scheme, rather than unduly enlarge or contract it by the substitution of more complex figures, especially if high and costly scaffoldings are necessary for the latter. Although not objectionable as far as the accuracy of the work is concerned, high structures are always very expensive, and should be avoided as much as possible. They are often necessary, however, to overcome obstructions, to elevate the line above the strata of highly heated and disturbed atmosphere near the surface of the earth, and may be less expensive than opening lines through timber, and the like.

Lines passing near the slope of mountains, ridges, or hills, or near any vertical surface, as the side of a building, or that pass through narrow avenues cut through timber, should be avoided, if possible, as they are particularly liable to lateral refraction, one of the greatest sources of error in triangulation. Any cause for undue atmospheric disturbance should be avoided, such as smoke from factories, and the like.

590. Base-line. Since the base-lines of a system of triangulation are of fundamental importance, all possible sites should be investigated, particularly in mountain regions, and also the scheme for connecting each with the main triangulation, so that the work of making the connection may be carried on simultaneously with the main work. This will obviate the necessity of reoccupying any of the stations merely for the base connection. The connection of the base with the main triangulation should be very carefully developed, as any extreme accuracy in the determination of the length of the base is soon lost unless the figures are well proportioned. Triangles containing angles less than about twenty degrees should rarely be used in any part of the triangulation, and never in the base connection, if they can possibly be avoided.

591. Records on Reconnaissance. The record of the reconnoiterer should be very complete, and can never be too exhaustive. Horizontal and vertical angles should be taken on all prominent

objects, mountain peaks, and the like, whether they are to be included in the scheme or not, as they are often invaluable for purposes of orientation. Observations for rough latitudes and azimuths should be made occasionally, and the magnetic bearing of prominent points noted. All difficulties of the country should be specified; the horizon sketched and described at each point, particularly every notch or opening through which more distant peaks or objects are visible. A rough topographical sketch should be made showing the main ridges, water courses, roads, trails, and habitations.

Comprehensive notes as to means of transportation, subsistence for man and animal, help, material, and accessibility, are invaluable. Remarks about weather and climatic conditions, cloudiness, and the like, are desirable, particularly in mountainous regions, where peaks are often in clouds for days at a time while elsewhere it is fairly clear.

592. Outfit and Instruments. The outfit and instruments required for reconnaissance will vary with the character of the country traversed and with the class of triangulation, but should always be as light and portable as possible.

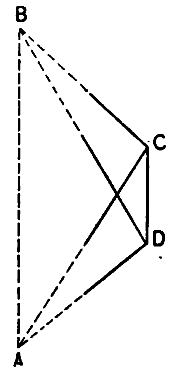
The following list of instruments is that recommended by the officers of the Coast and Geodetic Survey Report for 1893:

Two aneroid barometers; one 4-inch theodolite with vertical circle and tripod; one reconnoitering telescope of 3-inch aperture; azimuth compass and hand level; binocular; pocket box of drawing instruments, protractor, and scale; steel tape; two heliotropes for testing the doubtful intervisibility of stations; best maps of the country available; projection with river courses and known roads drawn thereon; small photographic apparatus with films; note and sketch books; a gradienter.

593. Intervisibility of Stations. It sometimes happens that two stations are separated by a strip of timber and are thus not visible from each other. Various methods may be used for determining where to cut the timber in order to open up the line, usually depending upon known angles and distances. If any three parts of a triangle in which this side enters are known, its direction may be

quickly computed; or if the latitude and longitude of the extremities of the line are known fairly well, the azimuth of the line may be computed from the formulas given on page 121. In case we have no data from other sources, we will usually be able to find two stations that are visible from each other and from which both extremities of the desired line are also visible, as shown in Fig. 435, A B being the line sought. By measuring the line C D and the angles at C and D, the other parts may be computed and the direction A to B, or B to A, obtained.

FIG. 435.



594. Height of Stations. When the country is flat or rolling, a very important problem is the determination of the height to which the stations must be elevated in order to overcome the curvature of the earth or to raise the line of sight above an intervening ridge or other obstruction.

The elevations of the stations and of the intervening obstruction must be determined as accurately as possible. If aneroid barometers are used the observations should be reduced accurately, as explained in Chapter XIV.

The following table will be found very useful in determining the heights necessary to overcome curvature. It is taken from the "Coast and Geodetic Survey Report" for 1882, Appendix No. 9, and contains the height of a tangent to the earth's surface at various distances from the point of tangency. The first column contains the distances in miles, the second contains the curvature of the earth corresponding to these distances—i. e., the height of the tangent from the curved surface at the various distances from the point of tangency; column three contains the effect of refraction, which tends to diminish the effect of curvature; and column four gives the combined effect of the two.

DIFFERENCE IN FEET BETWEEN THE APPARENT AND TRUE LEVEL AT DISTANCES VARYING FROM 1 TO 66 MILES.

Distance, miles.	Difference in feet for			Distance, miles.	Difference in feet for		
	Curvature.	Refraction.	Curvature and refraction.		Curvature.	Refraction.	Curvature and refraction.
1	0·7	0·1	0·6	34	771·3	108·0	663·3
2	2·7	0·4	2·3	35	817·4	114·4	703·0
3	6·0	0·8	5·2	36	864·8	121·1	743·7
4	10·7	1·5	9·2	37	913·5	127·9	785·6
5	16·7	2·3	14·4	38	963·5	134·9	828·6
6	24·0	3·4	20·6	39	1014·9	142·1	872·8
7	32·7	4·6	28·1	40	1067·6	149·5	918·1
8	42·7	6·0	36·7	41	1121·7	157·0	964·7
9	54·0	7·6	46·4	42	1177·0	164·8	1012·2
10	66·7	9·3	57·4	43	1233·7	172·7	1061·0
11	80·7	11·3	69·4	44	1291·8	180·8	1111·0
12	96·1	13·4	82·7	45	1351·2	189·2	1162·0
13	112·8	15·8	97·0	46	1411·9	197·7	1214·2
14	130·8	18·3	112·5	47	1474·0	206·3	1267·7
15	150·1	21·0	129·1	48	1537·3	215·2	1322·1
16	170·8	23·9	146·9	49	1602·0	224·3	1377·7
17	192·8	27·0	165·8	50	1668·1	233·5	1434·6
18	216·2	30·3	185·9	51	1735·5	243·0	1492·5
19	240·9	33·7	207·2	52	1804·2	252·6	1551·6
20	266·9	37·4	229·5	53	1874·3	262·4	1611·9
21	294·3	41·2	253·1	54	1945·7	272·4	1673·3
22	322·9	45·2	277·7	55	2018·4	282·6	1735·8
23	353·0	49·4	303·6	56	2092·5	292·9	1799·6
24	384·3	53·8	330·5	57	2167·9	303·5	1864·4
25	417·0	58·4	358·6	58	2244·6	314·2	1930·4
26	451·1	63·1	388·0	59	2322·7	325·2	1997·5
27	486·4	68·1	418·3	60	2402·1	336·3	2065·8
28	523·1	73·2	449·9	61	2482·8	347·6	2135·2
29	561·2	78·6	482·6	62	2564·9	359·1	2205·8
30	600·5	84·1	516·4	63	2648·3	370·8	2277·5
31	641·2	89·8	551·4	64	2733·0	382·6	2350·4
32	683·3	95·7	587·6	65	2819·1	394·7	2424·4
33	726·6	101·7	624·9	66	2906·5	406·9	2499·6

The table is computed from the following formula. Calling h the height in feet, s the distance in feet, and k the distance in statute miles, ρ the mean radius of the earth, and m the coefficient of refraction assumed at 0·070, its mean value:

$$\text{Curvature} = \frac{s^2}{2\rho}$$

$$\log \text{curvature} = 2 \log s - 7{\cdot}6209807$$

$$\text{Refraction} = \frac{m\, s^2}{\rho}$$

$$\text{Curvature and refraction} = (1 - 2\,m)\frac{s^2}{2\,\rho}$$

$$k = \frac{\sqrt{h}}{0{\cdot}7575} \text{ or } h = \frac{k^2}{1{\cdot}7426}$$

For distances less than thirty miles we may obtain the height within a foot from the following approximation:

The curvature in feet is equal to $\frac{2}{3}$ of the square of the distance in miles, diminished by $\frac{1}{7}$ for the effect of refraction.

Occasionally a line will pass over areas where the refraction is abnormally great. In such cases, if the observations are made at the period of maximum refraction it will not be necessary to build scaffolds as high as would be necessary for the average refraction.

If the coefficient of refraction for the region is known, it may be substituted for the value used in the computation of the above table.

The following examples will illustrate the method of using the table:

1. Suppose we have a line, A B, 25 miles in length, on a plain, required the height to which the stations must be elevated in order to make them intervisible. Since the line A B is on a plain, we have only the earth's curvature to consider. If we assume that the signals are to be of equal height, we can obtain what this height should be by entering the table with $12\frac{1}{2}$ miles and taking the quantity in column 4 corresponding to it, since the line of sight joining the two stations is tangent at its middle to the earth's surface. This quantity is 89·8 feet; hence, by erecting a signal at each station 89·8 feet in height, the tops of these two signals will be just intervisible with the ordinary refraction. If at station A we can only erect a signal 75 feet in height, we find from the table that the line of sight from its top would become tangent to the surface of the earth at a distance of a little less than $11\frac{1}{2}$ miles; and also from the table we find that B must be elevated about 106 feet, in order to overcome the curvature and refraction of the remainder of the distance, $25 - 11\frac{1}{2} = 13\frac{1}{2}$ miles. If we wish to raise the line of sight above the surface of the earth at the point of tangency, we must increase each of the signals by the amount that is desired, for ordinary cases.

2. "*Elevations required at given distances.*—If it is desired to ascertain whether two points in the reconnaissance, estimated to be 44 miles apart, would be visible one from the other, the natural elevations must be at least 278 feet above mean tide, or one 230 feet, and the other 331 feet, etc. This supposes that the intervening country is low, and that the ground at the tangent point is not above the mean surface of the sphere. If the height of the ground at this point should be 200 feet above mean tide, then the natural elevation should be 478, or 430 and 531 feet, etc., in height, and the line is barely possible. To insure success, the theodolite must be elevated, and at both stations, to avoid high signals.

3. "*To determine whether the line of sight between two stations would pass above or below the summit of an intervening hill, and how much in either case.*

h_1 = height of lower station.	d_1 = distance h_1 to h_2.
h_3 = height of higher station.	d_2 = distance h_2 to h_3.
h_2 = height of intervening hill.	

Example.

$h_1 =$ 600 feet.	600 feet strikes horizon at	32·3 miles,
$h_3 =$ 2000 feet.	$64 - 32{\cdot}3 = 31{\cdot}7$ miles	577 feet of elevation,
$h_2 =$ 1340 feet.	$31{\cdot}7 - 10 = 21{\cdot}7$ miles	270 feet of elevation,
$d_1 =$ 54 miles.	$2000 - 577$ feet	= 1423 feet,
$d_2 =$ 10 miles.	$\frac{64}{10} = 6{\cdot}4$ and $\frac{1423}{6{\cdot}4}$	= 222·3 feet,

and h or height of line at $h_2 = 1423 + 270 - 222{\cdot}3 = 1470{\cdot}7$ feet. Hence, the line passes 130·7 feet above the intervening hill and the stations are intervisible."

The question of intervisibility may also be determined by the following formula, in which the coefficient of refraction is reduced to 0·065 ·

$$h = h_1 + (h_3 - h_1)\frac{d_1}{d_1 + d_2} - 0{\cdot}5803\ d_1\ d_2.$$

Since the elevations vary as the square of the distance, it is evident that the least amount of construction is obtained by making the two structures of equal height when only the curvature is to be overcome. But if signals have to be elevated to raise the line of

sight above a ridge or other obstruction, then it is best to increase the height of the signal nearer the obstruction more than the other, and to make the difference between the two signals the greater the nearer the obstruction is to the signal.

595. Signals. The term signal, as used in triangulation, includes all structures and appliances employed as objects to designate to the observer the position of a station mark, and also includes all scaffolds, and the like, used to elevate either the object to be observed or the observing instrument.

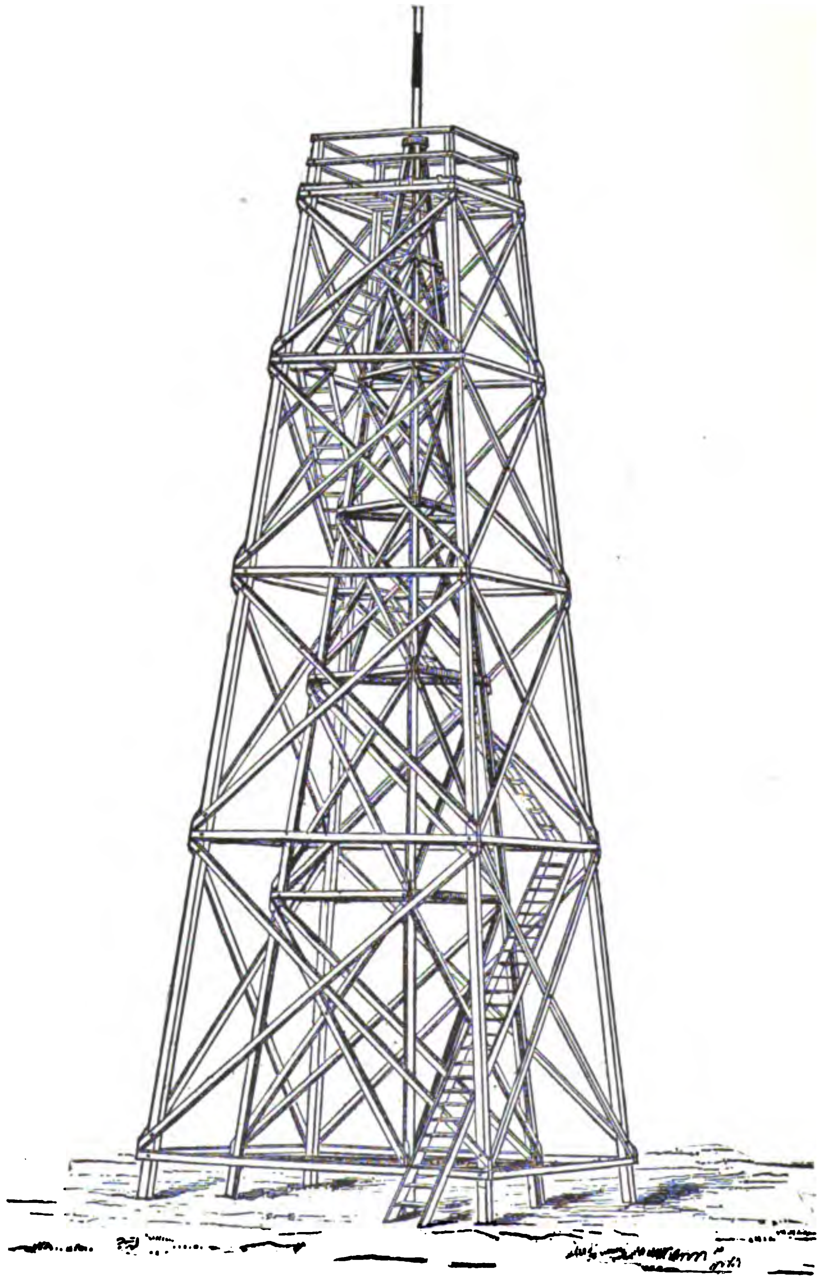
Observers have used for signals almost every class of object imaginable of sufficient prominence for identification, such as mountain peaks, headlands, rocks, trees, cairns, targets of various forms and construction, reflecting surfaces of tin, glass, etc., heliotropes, bonfires, rockets, lamps, magnesium and electric lights, etc. For supporting the instrument, mounds of earth, tree trunks, chimneys, lighthouses, scaffolds, stone or brick piers, etc., have been used.

Ordinarily the signals are of moderate height (say from 20 to 40 feet), and are rarely elevated to a greater extent except for the purpose of carrying the line of sight above the stratum of highly disturbed atmosphere near the surface of the earth, to maintain the general proportions of the triangulation, or to overcome some obstruction on the line.

All structures for elevating the instrument must be composed of two separate and entirely independent parts—the inner, known as the observing tripod, upon which the instrument is mounted, and the outer, called the scaffold. The observing tripod must be rigid and very stable, and is built of heavy timber. The scaffold may be built of lighter material, being merely to support the observer and carry the protection for the instrument from wind and sun. By wrapping the scaffold with canvas while observing, the tripod may be protected from the wind and sun as well as the instrument, thus practically eliminating one of the greatest sources of error in high structures, namely, the twist of the tripod due to unequal heating of its sides.

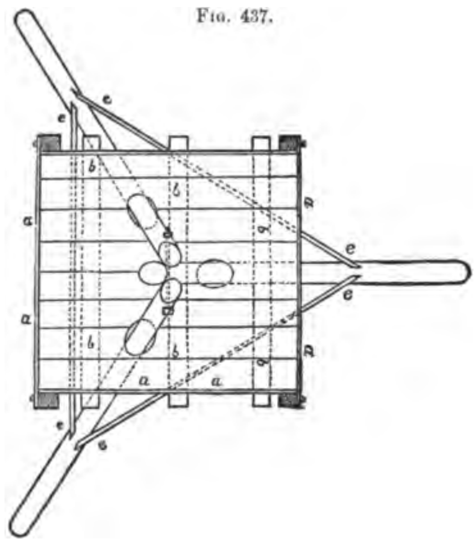
Fig. 436 illustrates the method of constructing high signals.

Fig. 486.



For the details of construction and outfit needed, see "Coast and Geodetic Survey Report" for 1882, Appendix 10, or 1893, pages 406–414.

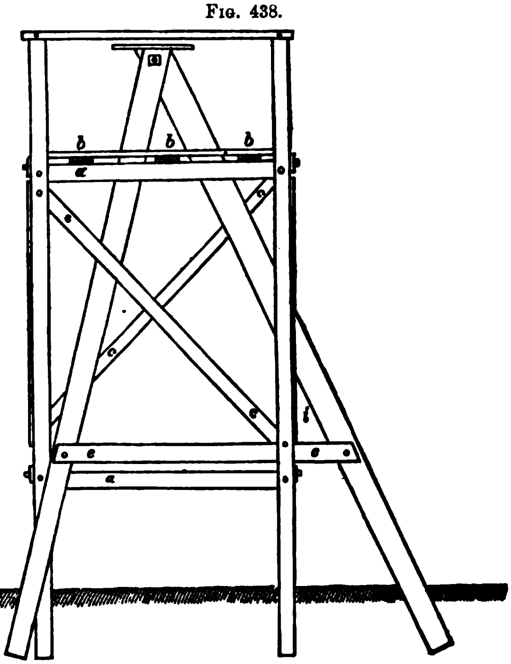
FIG. 437.



Structures similar to Fig. 436 have been raised to a height of 152 feet and used very satisfactorily.

Figs. 437 and 438 represent a convenient form of signal when the instrument has to be elevated only 12 or 15 feet. It is composed of two independent parts, tripod and scaffold, the same as with the larger structures.

FIG. 438.



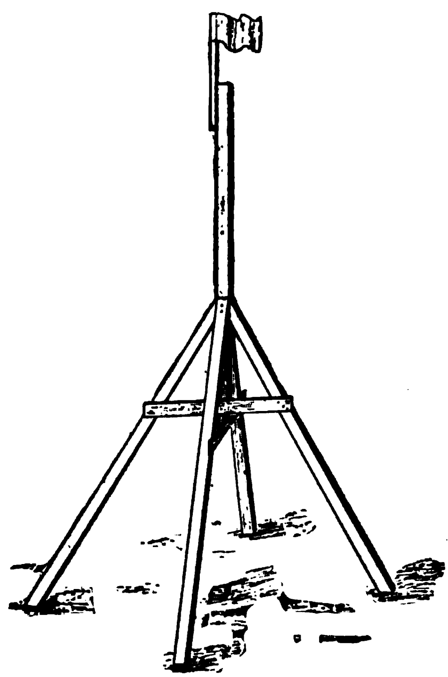
Figs. 439 and 440 show convenient forms of signals for tertiary triangulation or the smaller secondary. The pole, which is used to observe upon, is set high enough to allow the instrument to be placed underneath, when occupying the station, without disturbing the signal. Fig. 439 is used occasionally, as the four

braces make it more stable and rigid than the signal with but three braces, particularly when the ground is rocky and winds strong. The signal shown in Figs. 441 to 443 is also very convenient, particularly for low signals.

Fig. 441 represents the signal as framed on the ground; Fig. 442 shows it erected and ready for observation, its base being steadied with stones; and Fig. 443 shows it with the staff turned aside, to make room for the theodolite and its protecting tent.

The observable part of the pole above the braces should be long enough to be easily bisected, usually not less than $\frac{1}{1000}$ of the distance from which it is to be seen, or so it will subtend a vertical angle of at least half a minute at the instrument.

Fig. 439.



For lines less than five to seven miles in length, signal poles about three inches in diameter may be used satisfactorily with a fairly clear atmosphere and average telescopes. Four-inch poles will usually be visible for distances less than about 15 miles, and with a little better telescope than that used on the five- to seven-mile lines. If the atmosphere is clear, a four-inch pole may be seen at a distance of 20 miles, especially if the signal has a flag or target so that it may be readily identified. A fluttering flag is an excellent object to catch the eye when sweeping for a signal, but as observations are often made when there is no wind to make it flutter, a target fastened to the pole will answer the same purpose. A convenient form of target is to tack a strip

of board along one of the diagonals of a square piece of cloth, and then nail this strip to the signal pole, taking care to center it and also to make it perpendicular to the axis of the pole, then tack the other two corners of the square of cloth to the signal pole, so that this diagonal agrees with the center of the pole when seen by standing directly in front or behind it. This lozenge-shaped target makes an excellent object to point upon, and is only objectionable since it is eccentric, and consequently observations upon it must be corrected when it does not face directly toward the station occupied.

FIG. 440.

596. Requisites of observing Signals. A good signal must be clearly visible, rigid, capable of being accurately centered over the station mark and immovable when once in position, and of such shape that the center of the visible portion will always coincide

FIG. 441.

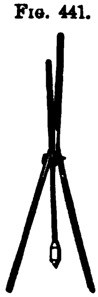


FIG. 442.

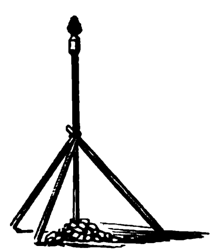
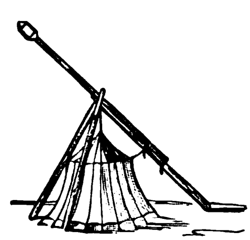


FIG. 443.



with the vertical of the station mark, whether in the sun or in the shade, and irrespective of the position of the observer.

One of the geodesist's greatest difficulties is to secure all these requisites. Cylindrical objects answer all these requirements excepting the last. Flat targets are good, especially if several are mounted one above another, each facing in the direction to be used. They are objectionable, however, owing to their shades and shadows, and the difficulty of accurately centering several at the same point.

When wooden targets are used they should be either painted (or covered with cloth) white for a dark background or black for a light background. If the lines are not very long, red may be used for a green background, although white is usually the best. On long lines, however, colors are rarely used, as they can not be distinguished from each other.

Conical reflecting surfaces, of tin or other materials, have been used considerably in the past, but are now almost abandoned, as there was always an uncertainty as to the exact part pointed upon.

Phase.—When these reflecting curved surfaces are used, the pointings are usually made upon the visible or bright part of the object, which varies with the position of the sun, and hence each pointing requires a correction for phase, as it is called, the amount of the correction depending upon the diameter of the object, position of the sun, and distance of the signal from the observer. The following formulas, derived directly from the geometrical relations, give the corrections necessary to reduce the direction to the center of the signal—i. e., correct it for phase.

When the bright line is pointed upon—i. e., the direct reflected rays only—the correction is

$$\pm \frac{r \cos. \frac{1}{2} a}{s \sin. 1''}.$$

When the whole illuminated part is pointed upon the correction is obtained from the following:

$$\pm \frac{r \cos.^2 \frac{1}{2} a}{s \sin. 1''}.$$

r is the mean radius of the object pointed upon,

s, its distance from the observer, and

a, the angle at the observing station between the sun and the object observed.

If the direction of the sun was not observed, it may be obtained from the following formulas, if the local time of making the observation was noted:

$$\tan. A = \frac{\tan. t \cos. M}{\sin. (\phi - M)}; \text{ where}$$

$$\tan. M = \frac{\tan. \delta}{\cos. t}.$$

δ representing the sun's declination,

t, its hour angle, or time from mean noon,

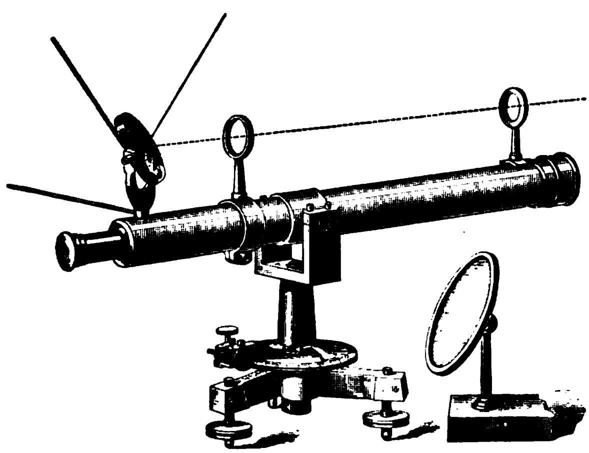
A, the sun's azimuth counted from the south point through the west, north, and east, to the south, and

ϕ, the latitude of the observing station.

Whether the correction for phase is positive or negative is readily seen from the relative positions of the sun and signal. For directions increasing from left to right, the correction for phase is minus when the sun is to the right of the signal and plus when it is on the left.

597. Heliotropes.—When it becomes necessary to use lines of greater length than 25 or 30 miles, the ordinary signal, no matter what its size, often becomes invisible; hence we must have

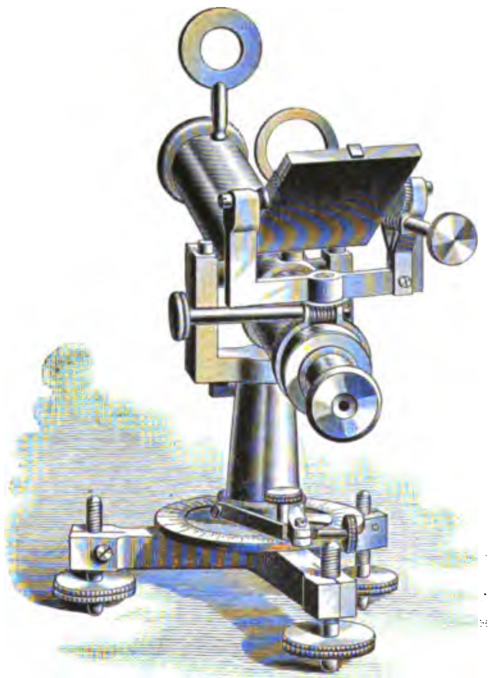
Fig. 444.



recourse to some other device. This is accomplished by the heliotrope, which is an instrument for reflecting the sun's rays. Fig. 444 shows such an instrument, consisting simply of a mirror

and directing arrangement attached to a telescope. The mirror is provided with two motions, one at right angles to the other, and so arranged that the center of the mirror will not be changed with reference to the telescope no matter how much the face is changed by means of the movements. For convenience in the field, each of these movements is made with a slow-motion screw (shown in Fig. 444[1]), so that the motion of the sun may be followed very closely. The directing apparatus consists of two rings so placed that the centers of their openings and the center of the mirror are all in one straight line. This central line is then made to point toward the station occupied by the observer, and the mirror so moved that it reflects the rays of the sun directly along this line.

FIG. 444[1].



In order that the man in charge of the heliotrope—or heliotroper, as he is called—may tell at a glance whether the rays are reflected directly along the central line or not, the ring next to the mirror has an opening slightly larger than that in the outer ring; hence, when the beam of light is central a narrow strip of light will be visible all around the opening of the outer ring, and all the heliotroper needs to do is to keep this ring of light concentric with the opening of the ring.

As it is usually desirable to make the pointing with the telescope rather than with the rings, we must so adjust the instrument

that the central line of the rings is parallel to the line of collimation of the telescope. After making the central line of the rings and mirror as nearly parallel with the telescope as possible by sighting along the side, direct it toward some distant and well-defined object (either by means of pointers centrally placed on the top of each ring and at equal distances from their centers, by stretching threads across the centers of the rings, or by any other method), and then move the cross hairs of the telescope, by means of their adjusting screws, until they bisect the same object.

When the sun is much behind the mirror on the heliotrope (which is showing in a given direction), an extra mirror, placed a little to one side and in front of the heliotrope mirror, is used to give a double reflection, and thus utilize the full size of the mirror.

As the sun is constantly changing its position, the heliotrope requires constant attention to keep the beam of light steady in one direction. Since the incident rays come from the sun, they must form a cone with the sun as a base, and hence the reflected rays must form a similar cone—i. e., one with a diameter of about 32 minutes. The base of this cone has a breadth of about 50 feet at the distance of a mile, consequently at a distance of 100 miles the heliotrope reflecting the light may be seen over an area, in a vertical plane, 5,000 feet in diameter. Hence the pointing of the heliotrope may be in error by 15′ and still be visible at the station occupied.

In order to secure images of nearly uniform brightness at all distances, it is necessary to vary the size of the mirror according to the length of line to be observed. For ordinary atmospheric conditions, and for distances greater than about 10 miles, the formula

$$x = 0{\cdot}046\ d$$

may be used, where x is the side of the square mirror in inches and d the distance in miles, or

$$y = \tfrac{3}{4}\ s$$

where y is in millimetres and s in kilometres.

The following table contains the length of side of square mirror for various distances:

Distance.	Side.	Distance.	Side.	Distance.	Side.
Miles.	Inches.	Miles.	Inches.	Miles.	Inches.
10	0·46	60	2·8	120	5·5
20	0·92	70	3·2	140	6·4
30	1·37	80	3·7	160	7·3
40	1·83	90	4·1	180	8·3
50	2·30	100	4·6	200	9·2

In the mountain regions in the western part of the United States much smaller mirrors have been used than those shown in the table, owing to the clearness of the atmosphere. The mirror should be as small as possible and still give a clear and distinct light, as it is a much better object to point upon when small and sharp then when large and very bright or diffuse.

The *Selenotrope* differs from the heliotrope only in the greater size of the mirror used, and is for the purpose of reflecting the light of the moon instead of that of the sun. It was first tested in 1885, on the primary triangulation along the thirty-ninth parallel of latitude, and gave excellent results.

In 1887, while occupying Mount Nebo, in Utah, with a view of testing the efficiency of the selenotrope upon long lines, mirrors 15 by 20 centimetres (6 by 8 inches), 20 by 25 centimetres (8 by 10 inches), and 30 by 46 centimetres (12 by 18 inches) were sent to Draper, Onaqui, and Ogden stations respectively—77, 113, and 156 kilometres (48, 70, and 97 miles) distant. The weather was unfavorable, excepting on two nights, when Draper and Onaqui were plainly visible in the illuminated field of the telescope, "distinct, steady, mere dots of white light, and of ideal perfection for precise pointing." Ogden for some reason was not seen, but whether due to the great distance or to some other cause is not known.

598. Night Signals. Numerous experiments have been made with night signals, using various kinds of lights. The magnesium light seems to have given the most satisfactory results, for the least expense, over long lines.

All who have made observations at night report favorably upon the method. The results on lines not exceeding 30 to 40 miles seem to be slightly better than those obtained during the day, and more free from lateral refraction. The atmosphere is nearly always

more steady during the night than during the day, consequently the signals are much better objects to point upon at night. On long lines (say 60 miles or more) night work is impracticable, as ordinary lights are invisible.

Heliotropes can not be used on cloudy days, when the atmosphere is most steady, and are objectionable on that account, as work is often greatly delayed by continued cloudiness. Night signals can be used at any time, and are particularly good after cloudy days when the atmospheric disturbance has been slight. Also, owing to the unsteadiness of the atmosphere during the day, observations on primary work can only be made at a short period just after sunrise and just before sunset; but with night signals they may be made at any time all through the night, thus expediting the work considerably.

Therefore, when extreme accuracy is desired it will usually be found preferable to make the observations at night, particularly where the atmosphere is very unsteady during the day, or where it is impracticable to use the heliotrope owing to continued cloudiness. The expense for night signals is but slightly in excess of that for heliotropes, and is usually more than compensated for in the shortening of the time necessary to occupy a station.

Another strong point in favor of night work is the fact that the refraction is much greater than during the day; hence in flat or gently rolling country, where signals must be constructed, they may be considerably lower if the observations are to be made at night, thus saving greatly in the expense of the work.

599. Marking Stations. "The main objects in marking a station are to secure its permanency and to render it easy of recovery. If the station is located on a ledge or rock not likely to be disturbed, a copper bolt with a cross on its top to mark the center, secured in a drill hole several inches deep, or two or three short bolts placed one over the other, so as to be more difficult of extraction, form a suitable station mark. Two or three arrows pointing to the center mark may also be cut in the rock. Where excavation is possible, there should be one mark at the surface and another buried three feet below the surface. For primary stations a stone em-

bedded in cement, with a copper bolt for center mark, forms the best sub-surface mark. If an observing pier or terminal of a base line is used, it should be built of stone, brick, or concrete, with a cross mark in the top and also one at the surface, and another below the ground, to indicate the center of the station, and two openings should be left in the pier at the base, at right angles to each other, to give access to the surface mark.

"In secondary and tertiary work a bottle, crock, or flowerpot filled with ashes may be used for a subsurface mark, with a stone and cross at the surface. Special conditions of soil require marks suited to their particular needs. In all cases there should be suitable witness or reference marks in addition to the central one, preferably grooves or drill holes (filled with sulphur) in adjacent rocks, to which the distances and bearings should be carefully noted.

"Experience has shown that stations are frequently lost by reason of the thoughtless meddling of ignorant and irresponsible persons, as well as by some whose cupidity had been excited by the material used in marking them, as, for instance, lead and copper in a country inhabited by Indians.

"Hence a good general rule is that stations should be marked by objects having little or no value. And it is important that the attention of the passer-by should not be attracted to the location of the station by reason of the prominence of the surface, reference, and witness marks. The aim should be to mark the spot in such a manner that there will be no difficulty for one who has its description to find it; but a casual observer should not have his attention attracted to it.

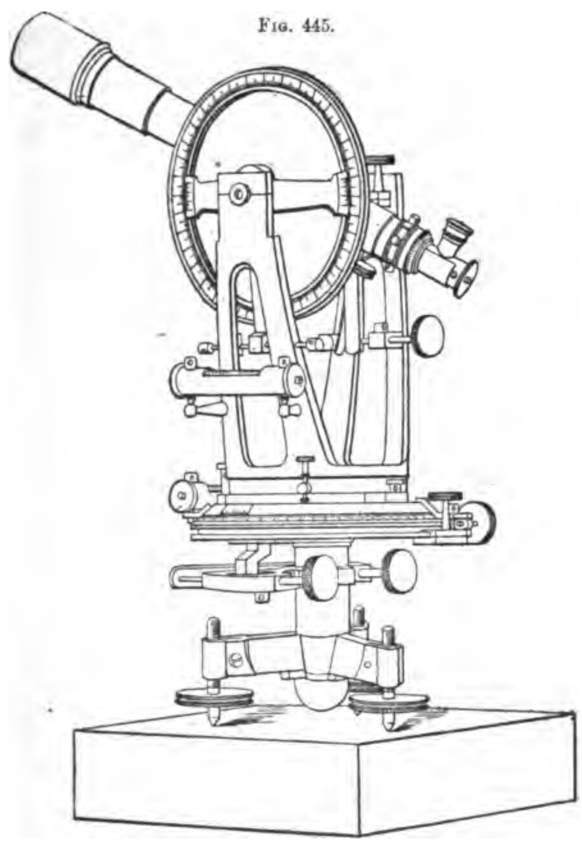
"The description of a station should include a topographical sketch of the ground and its approaches, a sketch showing the relative positions of the station mark and various points of reference, the best route by which to reach it from the nearest town, and any information which might prove of value to a party subsequently occupying the station, either in finding the station or in locating a camp or obtaining supplies. It is desirable that the name of the trigonometric station be a short one."

TRIANGULATION.

INSTRUMENTS AND METHODS OF OBSERVATION.

600. Instruments for Measuring Angles. Instruments for the measurement of horizontal angles have been constructed in almost all imaginable shapes and sizes, but since the introduction of the vernier and micrometer microscope for the determination of minute

FIG. 445.

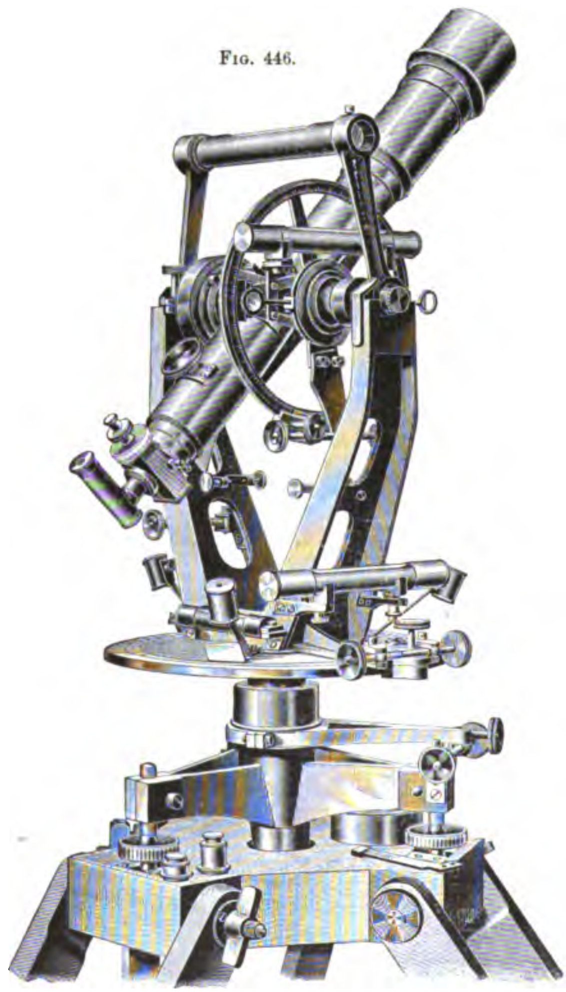


parts of the graduation, the size has been much reduced, until now the largest instruments used have circles about 20 inches (50 centimetres) in diameter; and even these are being gradually replaced by instruments with circles as small as 12 inches (30 centimetres)

in diameter, the means of graduating the circle becoming more nearly perfect. The shape, however, is still quite different in different instruments, although the same general principles are involved in each.

Fig. 445 represents a small theodolite, with 4-inch horizontal and vertical circles, used very much for reconnaissance purposes,

Fig. 446.

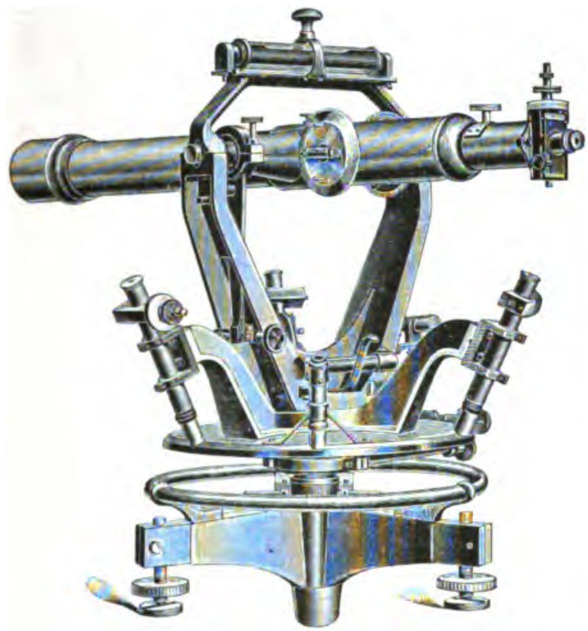


or in referring station marks, and also for making rough astronomical observations for latitude, time, and azimuth.

Fig. 446 is a theodolite, with an 8-inch horizontal circle, used in the measurement of horizontal angles, and for astronomical work.

Fig. 447 represents the latest form of 12-inch theodolite as used

FIG. 447.



on the Coast and Geodetic Survey. The greatest care possible was taken in the construction of its parts. Two small finders, or vertical circles, are attached to the telescope. Means are provided for illuminating the cross hairs of the telescope. A micrometer is attached to the eye end, and a diagonal eyepiece provided, so that the instrument may be used for astronomical work.

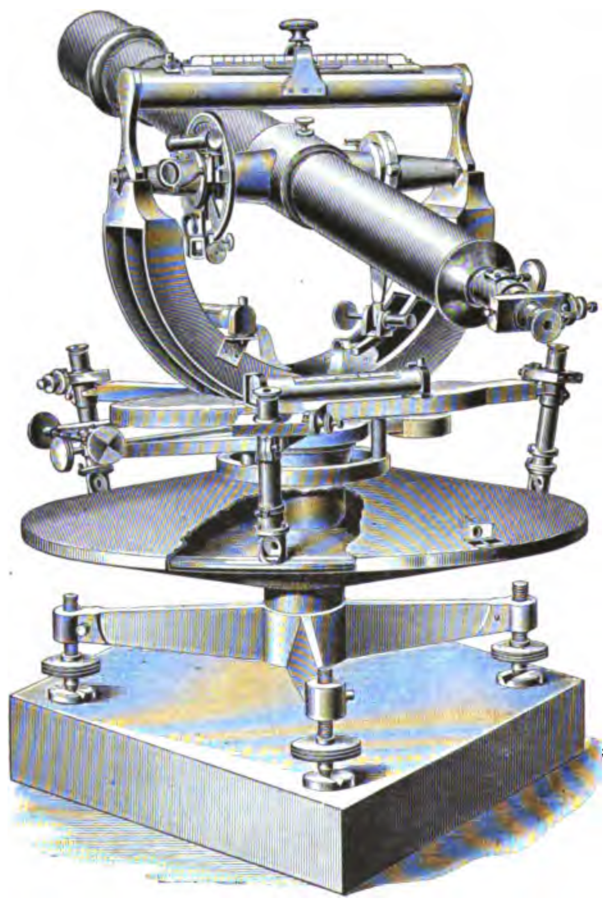
Fig. 448 shows a 20-inch theodolite, which is used for the same purposes as the 12-inch just described, and differs only in the details of the construction.

Various other sizes are used, depending upon the character of the work to be done, but they differ very little from those represented, excepting in the size and detail of construction.

All the best forms of theodolites rest upon three foot screws, and are as stable as possible. The vertical axes are as long as can be

obtained without sacrificing the stability and symmetry of the instrument. The magnifying power of the telescope is usually equal, if not superior, to the means for reading the circle. High-power telescopes are preferable to low, as they enable the observer to discern details and judge better the phase of the signals and steadiness

FIG. 448.



of the atmosphere. The magnifying powers of the telescopes of the larger instruments vary from about 50 to 100 diameters.

Instruments with circles of 10 inches diameter, and less, are nearly always read by means of verniers. On the smaller instru-

ments two verniers are usually provided, placed 180° apart, and three on the large ones, placed 120° apart.

Nearly all the larger instruments are read by means of the micrometer microscope, three of them being placed equidistantly about the circle. The smallest divisions used on verniers are usually 3″ or 5″ for the 10-inch circles, 5″ or 10″ for the 8-inch circles, and 10″ or more for 6-inch circles. 5″ divisions on the 8-inch circles and 3″ divisions on the 10-inch are too fine for the average instrument, however, and are only needed when the graduation is excellent, axes very true, and clamping arrangement almost perfect.

The eccentricity of the circle of a theodolite (i. e., the non-agreement of the center of the circle with the center of the vertical axes) and of the verniers or microscopes is eliminated as far as it affects the resulting directions or angles by reading any number of verniers or microscopes placed equidistantly about the circle. (See Chauvenet's "Astronomy," Vol. II, p. 37.)

601. Supports for Instruments. Instruments of the smaller class are usually mounted upon tripods with large heads. Each leg of the tripod is V-shaped—i. e., composed of two pieces, which pass on either side of a projection on the tripod head, and a bolt passed through all three fastens the leg to the head, while a nut on the end of the bolt forces them together, thus making a very rigid support.

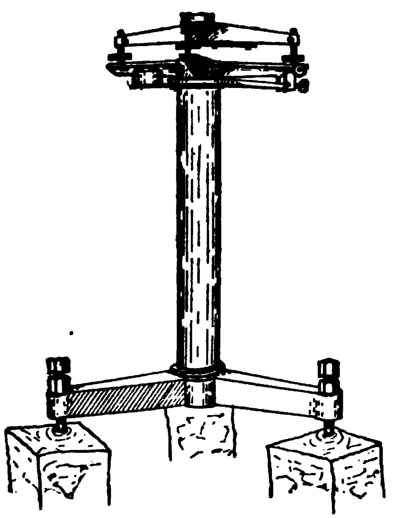
The larger instruments, especially those used on primary triangulation, must be supported in a better manner than even this form of tripod, as stability in the mounting is a primary requisite. Brick and stone piers have been used quite extensively, but the iron stand, similar to that shown in Fig. 449, is usually preferred, as it is not so clumsy as a masonry pier, and has a simple position circle attached to its upper part.

Theodolites are divided into two classes, repeating and nonrepeating or direction instruments.

The repeating theodolite is provided with two separate motions about vertical axes, one within the other. The plate upon which the graduated circle is placed and all the lower part of the instrument are attached to one axis, or lower movement, as it is called,

and the alidade is attached to the other axis, or upper movement. Hence, as the centers of these two vertical axes are identical, as nearly as it is possible to make them, the circle plate and all above it may be moved about the axis, or the circle may be clamped and the alidade alone moved. Each movement is provided with slow-motion screws and clamps. The repeater is used where it is desirable to get several measures or repetitions of an angle before reading the circle.

FIG. 449.



The object in making the circle movable as well as the alidade, is to enable the observer to lessen the effect on an angle of errors of graduation, reading, and pointing. Theoretically this method should give the better results, but practically the mechanical defects (such as slipping of the clamps, non-agreement of the centers of the two axes, etc.) vitiate in great measure the advantages gained, even when the graduation is irregular and read by means of coarse verniers. With such graduation and fine-reading verniers the gain from diminishing the effect of errors of graduation, reading, and pointing is usually more than balanced by the loss due to the slipping of the clamping apparatus, flexure, or lost motion.

The repeating theodolite is rarely used, excepting on tertiary or low-grade secondary triangulation, where it is desirable to complete the observations at a station as soon as possible. Even then, if the graduation is good and the verniers read finely enough, it may be better to use the method of directions, keeping the lower movement clamped, excepting to shift the zero of the graduation occasionally.

The non-repeater or direction theodolite is used on all refined work. It differs from the repeater in having but one movement, is ordinarily larger than the repeater, and is provided with some

means for reading the circle accurately, usually the micrometer microscope.

Some device for shifting the zero-point of the graduation on the limb, or graduated circle, must be used with all direction instruments. When the instrument rests upon a masonry pier a position circle must be provided for this purpose. It may be a very rough contrivance capable of being clamped in any position, but must be rigid and immovable when once so clamped. A simple position circle is that shown on the iron stand in Fig. 449. It consists of a long vertical axis with conical bearings at each end. The upper part has three grooved arms (for receiving the foot screws of the instrument) rigidly attached to it, and also a clamp with slow-motion screws. The vertical axis is a hollow cylinder about three inches in diameter and as long as the whole stand. It terminates in a cone at the bottom, with a small hole at the apex for use in centering over the station mark.

A very convenient method of centering the instrument, when the iron stand is used, is to fit its top with a piece of cardboard, making a small hole in the center of the axis of rotation; then, after leveling the arms upon which the instrument rests, bring the stand over the station mark and move it about until, by looking through the small hole in the cardboard, the center of the mark appears exactly in the center of the small hole at the bottom of the vertical axis. Then again test the level of the arms, and repeat the centering if necessary.

602. Cross Hairs. Various arrangements are used for fixing points of reference in the field of the telescope. All, however, are combinations of lines placed in the focal plane of the objective, and also in the focus of the eyepiece, since the two must coincide when the image of an object appears clear and sharp.

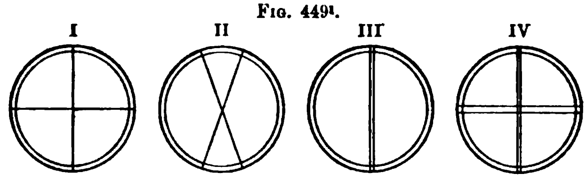
When only three or four lines are used the web of the spider is preferred. It should be fine, opaque, and black. The spider threads are fastened to a ring, or reticule, as it is called, by means of shellac, gum, wax, or anything that will make them stick.

When several lines are needed, particularly if it is desirable to have them placed equally distant, as in some kinds of astronomical work, they are usually cut on a thin piece of glass and blackened by rubbing a little powdered plumbago across them. This thin

piece of glass is then known as a diaphragm. The diaphragm is used as little as possible, since it always makes the image appear fainter, no matter how thin or clear it may be. The best theodolites are rarely supplied with glass diaphragms.

The following cuts represent the various arrangements of lines that are preferred in the different classes of theodolites:

FIG. 449[1].



I may be used in low-powered telescopes where the pointings are not very exact. II is preferred on theodolites used on tertiary triangulation, or occasionally on the secondary triangulation, particularly in the repeating instrument. III is used on telescopes or microscopes, where lines are to be pointed upon, as in the microscope for reading the graduated limb of the theodolite, or in the telescope for pointing upon a long target, and the like. IV is used in the high-power telescopes of the best theodolites, where the objects pointed upon are heliotropes or targets. The angle subtended by the vertical lines is about 15′ for a telescope that magnifies sixty or seventy diameters, while that between the horizontal lines is usually much greater. The latter serve merely to limit the parts of the vertical lines that are to be used, so as not to introduce errors due to any irregularities in these lines.

603. Adjustments of the Theodolite. The adjustments of the various parts of the theodolite are similar to those of the engineer's transit, but are made in a more direct manner.

Level adjustments.—All fixed levels must be treated exactly as with the transit—i. e., the bubble is brought to the center when parallel to one of the arms of the three foot screws (or perpendicular to this—i. e., parallel to the line joining two of the screws); then the instrument is revolved 180° in azimuth. If now the bubble is not in the center, correct half the error by means of the adjusting screw of the level, and the other half by means of the foot screws. Repeat until the bubble remains in the center for all positions of the instrument.

A striding level is adjusted by moving its adjusting screws until it reads the same after reversal as before—i. e., if it rests upon an axis, either horizontal or inclined, the bubble must be the same distance from the center after the level has been turned end for end as before. If it is not, correct half the difference by means of the vertical adjusting screws, and repeat until the adjustment is perfected.

The wind, or lack of parallelism between the level vial and the line joining the centers of the points of supports, is tested by tipping the level from one side to the other and noting whether the bubble changes its position or not. If it runs toward the right, for example, when tipped toward the observer, then the right-hand end of the vial must be moved toward the observer by means of the lateral adjusting screws of the level, until the bubble remains unchanged, whether the level is tipped to one side or not, while resting on cylindrical pivots.

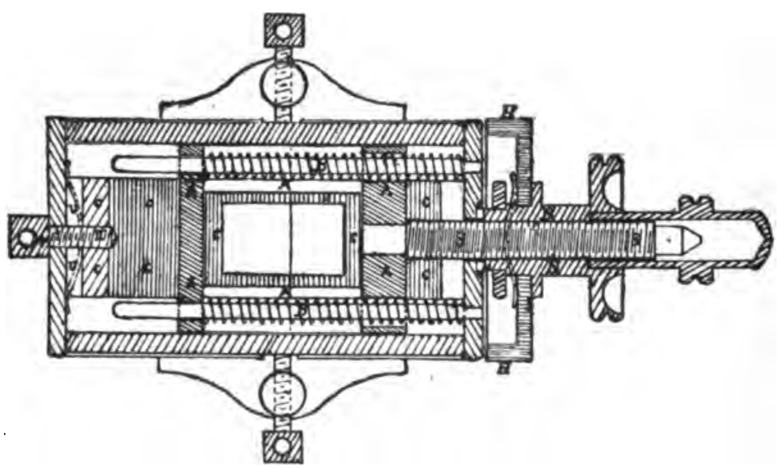
To make the Revolving Axis of the Telescope Horizontal.—After adjusting the levels, make the vertical axis approximately vertical with the small fixed levels, and then perfect the verticality by means of the striding level—i. e., move the foot screws until the bubble remains unchanged while the instrument is moved in azimuth. If now the striding level is in adjustment, any deviation of the bubble from the center is due to the inclination of the horizontal axis of the telescope, and must be corrected by bringing the bubble to the center, by raising or lowering the pivot at the end of the axis with its adjusting screws.

Collimation.—Having perfected the adjustments just described, particularly the horizontality of the revolving axis of the telescope, point the telescope on some sharp, well-defined object (not less than a mile away with the larger telescopes, so as to use the focus that will be needed for all distant objects), clamp the circle firmly, and then lift the telescope from the wyes, invert it, replace it in the wyes, and note whether the object previously pointed upon is still bisected or not. If it is not bisected, correct half the error by means of the cross hair adjusting screws on the sides of the telescope near the eye end. If the telescope is inverting, the cross hairs must be moved in the direction opposite to that apparently required.

FILAR MICROMETER.

604. The filar micrometer, or micrometer, as it is usually called, is used for the accurate measurement of small distances, or angles when they are so small that the chord and arc subtending the angle may be considered equal. It consists of a device for measuring accurately the motion of a cross hair or thread across the field of a microscope. Fig. 450 shows the principal parts of the device. The cross hairs are stretched across a frame or reticule, A A, which has a fine, smoothly cut screw, S, rigidly attached to it. The reticule moves between guides on the small metal box that incloses it. The screw projects beyond the box and works through a nut, N, on the outside. This nut bears against the end of the box, and hence the screw (and reticule to which it is attached) must move whenever the nut is turned. Two spiral springs, B B, inside of the box keep

FIG. 450.



the reticule constantly pressed away from the nut, and thus prevent the introduction of errors due to lost motion in the screw or lack of parallelism in the opposite faces of the threads of the screw or nut. The nut carries a graduated head, H (preferably of zylonite or some white material), for reading fractions of a turn of the screw.

Another frame, C C C, is usually placed in front of the movable reticule to carry a comb or rack arrangement for use in denoting the number of whole turns of the screw (the fractions being obtained from the graduated head outside), and also to carry a fixed reference line, or a glass diaphragm when used for astronomical work. This frame is adjustable by means of a screw, T, and spring, V V, on the end opposite to the nut or micrometer head. The faces of the fixed and movable frames should be as close together as possible, so that the two sets of lines may be placed in the same focus, and thus used interchangeably without refocusing.

The micrometer box is so placed that the thread or wire is in the common focus of the eyepiece and objective of the microscope, and also so that the motion of the reticule is perpendicular to the line of sight of the microscope and parallel to the line to be measured.

When the micrometer is used on the reading microscope of the theodolite, two parallel threads, stretched across the reticule, are usually preferred, the distance between them being slightly greater than the width of the graduation marks to be pointed upon, so that the eye will see a very narrow strip either side of the mark, and thus make an accurate bisection. The $\times$ has been used, and also a single line, for this purpose, but are not liked so well, for the eye discovers slight errors most readily with the parallel lines.

The micrometer head on the reading microscope of the theodolite is divided into 60 equal parts, and the screw and microscope so designed that one turn is one minute, and hence each division of the head one second of arc as nearly as may be. When the micrometer is used for other purposes, however, the head is divided into 100 equal parts, the decimal system being preferred unless the reading of the head gives the angle directly.

The division spaces of the limb of the theodolite where micrometer microscopes are employed are usually $5'$ between successive graduation marks; hence the micrometers must have a range of at least two spaces, or ten turns of the screw. If the divisions of the head are numbered so as to increase when the threads or lines are moved against the graduation of the limb or backward, then when the threads bisect the first graduation mark back of the fixed thread

(or zero of the micrometer) the reading is the distance of the fixed line beyond that graduation mark, provided the fixed line has previously been adjusted so that the micrometer reads zero when pointing upon it. If now the micrometer be run in the opposite direction until the first mark in advance is bisected, it should require just five turns of the screw to move the thread from one bisection to the next, if the graduation, adjustment of the micrometer, and pointings are perfect. Practically, however, it is not easy, nor is it essential, to so adjust the microscope and screw that five complete turns will move the threads from one bisection to the next, owing to inequalities in the width of the spaces (i. e., errors of graduation), or variations in the distance between the plane of the graduation and the micrometer thread. The latter may be due to the expansion or contraction (for changes of temperature) of the metal composing the microscopes or their supporting arms, or the plane of the graduation may not be parallel to the plane of motion of the micrometer thread, either in its box or when the whole micrometer is moved horizontally with the alidade.

In order that the five turns may be made to average very nearly the five minutes of arc, the microscope is adjustable, so that the image may be increased or diminished at will, by raising or lowering the microscope on its supporting arm, and, of course, readjusting the focus. After the most careful adjustment, however, the two readings will differ from five complete turns by a small quantity which is called the "run of the micrometer." If it were not for this "run" only a reading on one of the graduation marks would be necessary. As it is, readings on both graduation marks are always made and their mean, after applying the correction for "run," taken as the true reading of the micrometer.

If the screw threads are not uniform then each turn may have its value determined and tabulated, so that each reading may be corrected for the particular part used. Ordinarily, however, a screw that is not sufficiently uniform should be discarded. The graduation is so nearly perfect that by taking the mean of a small number of spaces distributed about the circle the effect of the irregularities is practically eliminated. Hence the microscopes are so adjusted that the average "run" is about zero with a normal temperature, and

then an average "run" for all the microscopes obtained for some short period, as a day, or preferably during the periods of uniform temperature, and each reading during that period corrected for the effect upon it of this average "run."

Determination of the "Run."—Let D represent the mean value in micrometer divisions of the spaces between consecutive graduation marks about the circle. With 5′ spaces and micrometer head graduated to 60 divisions, $D = 300^d$.

$\triangle$ is the number of divisions in any one space—i. e., $\triangle = D + r$ where

r is the "run" of the micrometer in that space.

b is the back reading of the micrometer, or the distance in terms of the micrometer screw, of the zero line of the micrometer from the graduation mark last passed.

f is the forward reading of the micrometer, or its reading when pointing upon the graduation mark next in advance of the zero line of the micrometer. Since f is obtained by moving the micrometer head against its graduation, the actual number of divisions passed over between the zero line and graduation mark is equal to $(D - f)$.

Hence $\triangle = b + (D - f)$, but $\triangle = D + r$, whence $r = b - f$, which corresponds with the definition of the "run" previously given.

It is evident that when the zero line of the micrometer falls halfway between the two graduation marks, the mean of the two readings must be free from the effect of "run," as the overrun or underrun is the same in each direction. If the zero line almost coincides with the graduation mark, it is also evident that the mean of the two readings must be corrected for one half the run in the whole space, this correction having one sign at the beginning of the space and the other sign at the end. Any intermediate readings must be corrected relative to their positions in the space.

When the increasing numbers on the micrometer head correspond to a backward motion along the limb, $b - f$, positive, indicates an overrun—i. e., that the value of one turn is a little less than 1′.

Since the run of the micrometer is considered uniform over

the part of the screw used, the correction to either b or f must be such a proportion of r as the space traversed by the micrometer, in each case, is a part of the total space traversed between the two graduation marks. Hence the correction to b is

$$c' = -\frac{b\,r}{\Delta} \qquad \text{or} \qquad c' = -\frac{b\,r}{D}$$

since Δ differs but little from D. Also the correction to $D - f$ is

$$c''' = -\frac{(D-f)\,r}{D}$$

and the correction to f is

$$c'' = +\frac{(D-f)\,r}{D}$$

If now we combine these two corrections by taking their mean we will obtain the correction (c) to be applied to the mean of the back and forward readings in order to free it from the error of run:

$$c = \frac{c' + c''}{2} = \frac{(D-f)\,r - b\,r}{2\,D} = \tfrac{1}{2}\,r - \frac{m\,r}{D} \qquad [50.]$$

where $m = \frac{b+f}{2}$. Finally, if M is the corrected reading

$$M = m + \tfrac{1}{2}\,r - \frac{m\,r}{D} \qquad \text{or} \qquad M = m + c \qquad [51.]$$

The member $\frac{1}{2}\,r - \frac{m\,r}{D}$, or $\left(\frac{1}{2} - \frac{m}{D}\right) r$, may be tabulated for varying values of m and r, and the corrections (c) obtained directly from the table. Such a table, for instruments graduated to 5′, may be found on page 385, "United States Coast and Geodetic Survey Report" for 1884.

In practice, the effect of the error of run on the final directions is always very small, hence it is not necessary to consider each micrometer by itself, but all the back micrometer readings for each pointing of the telescope are combined, as also all the forward readings, and these means treated the same as though they were obtained from a single micrometer. For example, if A, B, C are the three back readings of the three micrometers of a theodolite for any one pointing, then $\frac{A + B + C}{3} = b$, and $\frac{A' + B' + C'}{3} = f$,

where A′, B′, and C′ are the corresponding forward readings. b and f are to be used as in equation [50], and according to the definitions on page 47.

As the errors of graduation, pointing, reading, etc., must appear to a certain extent in the individual run determinations, it is usually better to get the mean run for such periods when the temperature is fairly constant, and then correct each value of m during this period for this mean run instead of the run given by that particular reading; for these other errors are more or less accidental, and consequently the greater the number that we combine together the greater is the probability that they will counterbalance each other and thus leave only the actual run to be corrected for.

Page 58 illustrates the method of recording the three micrometers of a theodolite. The minutes are obtained by adding the turns of the back reading to the value of the graduation mark pointed upon. The quantities in the columns headed "back" and "forward" are the actual readings on the head of the micrometer indicated for the back and forward pointings respectively. Underneath are the means of the readings of the three micrometers, or b and f respectively. On the right hand is the difference, $b - f$ (or run), the mean of the two means $\left(\frac{b+f}{2}\right)$, the correction to this mean (c), and finally the corrected value of the observation.

Suppose we have been able to get 120 such observations during a period when the temperature range was not very great. We first construct a table giving the values of c for varying values of m and the mean value of r obtained from the 120 observations. Suppose the algebraic sum of the 120 r's is $-$ 70·5 divisions, then the mean r is $-\frac{70 \cdot 5}{120} = -0 \cdot 5875\ d.$ Substituting this value for r in equation [50] and transposing, we get

$$c = -0 \cdot 5875 \frac{150 - m}{300}$$

Since ordinarily only tenths of divisions are read, we may construct the working table giving the limits of m between which the correction c is the same, and arrange it so that we have the limits

for each unit of the correction. Substituting for c, in the last equation, 0·1, which is the unit that we wish to use, we get

$$0{\cdot}1 = -\,0{\cdot}5875\,\frac{150 - m}{300}$$

whence
$$150 - m = -\frac{30}{0{\cdot}5875} = 51{\cdot}1. \qquad [52.]$$

which is the amount m may change before c will change 0·1. We keep 150 in the first member, as all corrections are referred to the center of the graduation space, or $m = 150^d$, that being the point at which the correction is zero.

Since the correction becomes 0·1 as soon as it is greater than 0·05, we have the first limit ½ of 51·1, or 25·5, as the distance either side of the center of a space (150^d or $2^t\ 30^d$) that the correction remains zero, for this particular value of the run. The other limits are obtained by adding or subtracting $51^d{\cdot}1$ to the limits previously found, thus obtaining the following table:

LIMITS FOR m.	CORRECTION c.	LIMITS FOR m.
$-0^t\ 28^d{\cdot}8$		$5^t\ 28^d{\cdot}8$
	$-0{\cdot}3+$	
0 22·3		4 37·7
	$-0{\cdot}2+$	
1 13·4		3 46·6
	$-0{\cdot}1+$	
2 4·5		2 55·5
	$-0{\cdot}0+$	
2 30·0		2 30·0

From equation [52] we see that the correction must take the same sign as the run for values of m less than 150^d or $2^t\,30^d$, and for m greater than 150^d it must take the opposite sign.

If now we wish to obtain the correction for the observed reading 10° 21′ $32^d{\cdot}3$, we enter the table with $1^t\ 32^d{\cdot}3$, and find it falls between $1^t\ 13^d{\cdot}4$ and $2^t\ 04^d{\cdot}5$, hence the correction is $-0^d{\cdot}1$, and the corrected reading becomes 10° 21′ 32·2″. It will be noticed that the 32·3 in the uncorrected reading is given in terms of the micrometer, but after applying the correction for run it is changed to seconds of arc. If the reading is 10° 19′ $46^d{\cdot}6$, we find from the table the correction $+\ 0^d{\cdot}3$, provided the 15′ and 20′ graduation marks were pointed upon. If the pointings were made on the 20′ and 25′ marks,

then the table must be entered with $-13^{d}\cdot4$, and we obtain the correction $-0^{d}\cdot3$, or with the opposite sign.

605. Method of Observing. There are two general methods of making observations, viz., the method of repetition, where the repeating theodolite is used, and the method of directions, where the instrument is a direction theodolite.

In the first method each angle is measured independently by repeating it a number of times by successive additions on the limb, and then reading this multiplied angle, which, when divided by the number of repetitions, gives the observed value of the angle.

In the second method each station is pointed upon in succession and the circle read at each, thus giving the direction of each station with reference to the initial station or reference mark.

Method of Repetitions.—The programme usually preferred in the method of repetitions may be briefly stated as follows, care being taken to touch or disturb but one clamp and corresponding tangent screw at a time:

Programme.

First adjust and carefully level the instrument, and then with telescope direct,

1. Set on the left station, and read all the verniers.
2. Unclamp above and set on right station.
3. Unclamp below and set on left station.
4. Unclamp above and set on right station.
5. Unclamp below and set on left station.
6. Unclamp above and set on right station.
7. Unclamp below, reverse telescope, and set on left station.
8. Unclamp above and set on right station.
9. Unclamp below and set on left station.
10. Unclamp above and set on right station.
11. Unclamp below and set on left station.
12. Unclamp above and set on right station, and again read the verniers.

This makes six repetitions of the angle, three with the telescope direct and three with it reversed, thus eliminating any error due to lack of adjustment of the line of collimation, or inclination of the

revolving axis of the telescope, which would otherwise enter whenever the two stations are at different elevations.

If the telescope does not transit, it may be lifted from the wyes, inverted, and replaced after revolving the instrument 180°, thus accomplishing the same thing as by transiting. As there is liability to disturbance while reversing the telescope by lifting it from the wyes, it may be better to make a complete set of six repetitions with telescope direct and then another with telescope reversed, the mean of the two sets being the same as the mean of two sets by the method outlined above, in so far as the elimination of the errors of adjustment are concerned.

It may become advisable occasionally to make less than six repetitions in a set, although fewer sets of six repetitions each are usually preferable to fewer repetitions in a set, the total number of repetitions being the same.

All repeating instruments are lacking in stability, the effect of which, together with the lost motion in the numerous movable parts, and particularly in the clamping arrangement, can only be eliminated by most careful manipulation and the adoption of a method of observation which will make them always of the same sign.

Experience has shown that this is best accomplished by always moving the alidade in one direction, making first one set of observations on the angle itself and then a set upon the explement of the angle, and correcting each by half the error of closure, or difference between their sum and 360°. This difference has been found constant, within the probable error of pointing and reading, for any particular condition of the instrument irrespective of the size of the angle. And since angles measured according to this method, under conditions which give closing errors of wide range, show a close accord when corrected by half their closing errors, it seems probable that the method largely eliminates errors from this source.

It is desirable to secure an equal number of measures with the telescope direct and with telescope reversed, especially when the stations are at different elevations, in order to eliminate the effect of errors of collimation and inclination of the revolving axis of the telescope.

The effect, upon a horizontal angle, of an error of 10″ in the collimation of the telescope, with elevation of object 3° (or of 1′ with elevation of object 1°), is less than 0·01″, hence it may always be kept very small.

The effect of an error in the horizontality of the revolving axis of the telescope is in general much larger, as it depends upon the tangent of the angle of elevation. Thus, for an inclination of 10″ in the axis, and elevation of the object 1°, the observed angle is in error 0·2″.

The effect of an error in the verticality of the theodolite axis can not be eliminated by any method of observation. The amount for any angle depends upon the relation of the angle to the vertical plane of the inclined and vertical axes and upon the amount of the inclination.

The number of sets of six repetitions required for the determination of an angle will vary with the class of work, accuracy desired, and character of instrument used. For tertiary work, two sets, or even two sets of three repetitions each, may be sufficient to obtain the required accuracy, while for secondary work it may be necessary to take six sets of six repetitions each, or even more. It is always necessary to take at least two sets in order to close the horizon, as explained above, and get a check upon the work.

Number of Angles to be measured at a Station.—In general there should be a check upon every angle at a station in addition to that mentioned in the last paragraph. Although, for the highest degree of accuracy, all the combinations of angles possible at a station might be measured, it is rarely done, especially where the number of angles is large, as the increase in the accuracy of the result is not commensurate with the time and labor expended. It is preferable to measure only those angles which will be used in the triangulation; and this consideration should have some weight in selecting those sum angles which are to be used as checks, and which will equalize the number of pointings as nearly as may be.

Method of Directions.—In this method, as mentioned above, pointings are made on each station in succession around the horizon, reading all the micrometer microscopes or verniers each time. If another pointing be made on the initial station after passing around

the entire horizon, there will usually result a small difference, which may be distributed equally among the pointings. This method, however, is rarely followed, as such an adjustment ordinarily adds but little to the accuracy of the resulting directions. The most common method is to observe each station in succession by moving with the graduation and with telescope direct, and then, after making the pointing on the last station, reverse the telescope and repeat the pointings in the reverse order—i. e., observe, with telescope direct, A, B, C, etc., to N, then, with telescope reversed, observe N, etc., to C, B, A.

The combination of these two sets of readings to a mean is known as a *series*, and the mean of the two readings on each station, after subtracting the mean of the two readings on the initial station, is the *direction* of the station with reference to the initial station, as obtained from this series. By reversing the telescope each series is freed from the effect of errors in the adjustment of the line of collimation and inclination of the revolving axis of the telescope. By reading both to the right and the left, each series is freed from the effect of any steady or uniform twist, such as occurs on high observing tripods. This last has been observed to be as great as 1″ in a minute of time on a 75-foot signal.

This method, therefore, eliminates most of the instrumental errors of any consequence, excepting the errors of graduation. If it becomes necessary to make only a few observations, the errors of the graduation may be carefully determined, and tables prepared giving the correction for each part, but this method is not generally used. Owing to the varying atmospheric conditions, it is nearly always possible, with the best instruments, to get enough observations distributed about the circle to eliminate the effect of the errors of graduation before the atmospheric effect is sufficiently annulled. Hence, to eliminate the effect of errors of graduation, series are observed in a number of positions uniformly distributed about the limb. This number should be such that two readings on any one station will not fall on the same part of the circle.

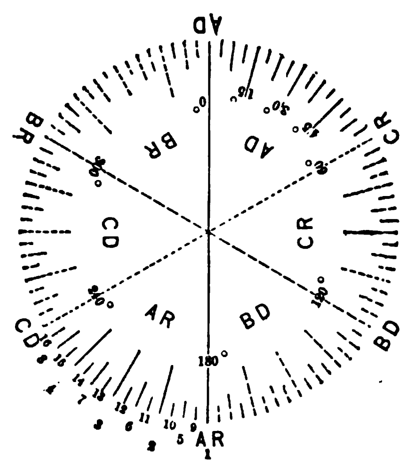
Number of Observations necessary.—The United States Coast and Geodetic Survey find that about thirty-one series are probably needed to secure the desired accuracy on the best class of work with

the large instruments, and that these series may be taken in as many positions as the observer should judge necessary, taking into account the character of the work and instrument used. In the past, twenty-three positions have been used, but lately seventeen has been the limit, the intention being to obtain two series in each.

Positions.—Some observers divide the entire circumference into as many equal parts as the number of positions they wish to observe, then (if seventeen positions are to be observed) shift the zero of graduation of the limb by $\frac{1}{17}$ of 360°, or 21° 10′ 35″ for each successive position, by means of the position circle. This corrects the eccentricity of double vertical centers also, which is especially important at the ends of a base or short lines.

Another method is to divide half the space between two microscopes by the number of positions, and shift the zero by this amount, which covers the whole circle, as shown by the accompanying diagram (Fig. 450^1). The full lines represent the positions microscope A falls upon when pointing on the initial station direct and reversed. The broken lines represent the corresponding positions of B, and the dotted lines those of C.

Fig. 450^1.



Still another method is to take a small number of positions uniformly distributed over the space of half the distance between the microscopes, as, say, 1, 2, 3, 4, in Fig. 450^1, then repeat with the same number of positions, but with a different initial reading of the graduation, and subdividing the former spaces, as 5, 6, 7, 8. Additional groups may be added, if greater accuracy is demanded, by further subdivision, as 9, 10, 11, 12, 13, 14, 15, 16, etc. The advantage claimed for this procedure is the easy comparability

of the results of the series making up the groups, which would be of great value if the accuracy depended less upon the atmospheric conditions, and more on the number of observations and positions.

Initial Station.—When the lines of the triangulation are so long that heliotropes are needed, it is impracticable to use any of the regular triangulation stations for the initial station, as it is liable to be cut off by clouds, or other causes, just when most needed, thus retarding the work and vitiating the accuracy of the results. Hence a target is set on some point distant between 1½ and 10 miles, which is used as the initial station, and called the reference mark, or simply the mark. This mark should be in the horizon (or at the same elevation as the majority of the stations to be observed, if possible), and preferably toward the north, for convenience in measuring the astronomical azimuth, and also on account of the shadows. It may be placed within 1½ mile, provided its image in the telescope is clear and sharp for the focus that must be used for the other stations; but if it is more than about 10 miles away, it is liable to give the same trouble as one of the triangulation stations.

When parallel lines are used in the telescope, the best form of target is a long, narrow strip of black with a strip of white on each side. The black strip should be a trifle smaller than the opening between the lines, so that a narrow strip of white will be visible on each side of the black strip and inside of the spider lines of the telescope. The width of the black strip necessary for a certain distance may be easily computed, if the angular opening between the spider lines is known, by the relation that a second subtends a foot at a distance of 40 miles. For example, if the angular opening between the spider lines is 15″, and distance from instrument to the point where the mark is to be placed is 3 miles, and we want to have a second clear on each side of the black for the white to show, then we have the proportion $(15'' - 2'' = 13'')\ 40 : 13 :: 3 : x$, whence $x = 0{\cdot}975$ feet, the width of black necessary to subtend an angle of 13″ at a distance of 3 miles.

Time to Observe.—It is well known that, owing to changing atmospheric conditions, there is considerable refraction laterally as well as vertically. Hence it is necessary, for the best work, to distribute the observations over several days, and at different times

during the day, say forenoon and afternoon, under varying atmospheric conditions, and refrain from observing under any manifestly unsuitable or doubtful conditions.

In no case in primary and secondary triangulation should the observations be finished in one day, but several days, embracing observations as uniformly distributed between the forenoon and afternoon as possible, should be devoted to them, in order to eliminate as much as possible this lateral refraction, which has occasionally been experienced to the extent of several seconds.

In mounting the instrument, regard should be had to proper shelter for it as well as for personal comfort while observing. Pointings should be made as rapidly as possible consistent with a clear bisection of the signal. Particular attention should be paid to the centering of the instrument and signals exactly over the station marks. When it is necessary to mount the instrument or signals eccentrically, the distance from the station mark and direction of the line joining the station mark and eccentric mark should be carefully measured and noted so that the eccentrically observed directions may be reduced to the center of the station mark.

RECORD FOR HORIZONTAL ANGLES.

Station, Omega. State, Colorado. County, Alay. Date, June 7, 1895.
Observer, O. K. Instrument, 8-in. Theodolite, No. 52.

OBJECTS OBSERVED.	TIME. (h. m.)	TEL. D OR R.	REP'S.	ANGLE.	A	B	C	MEAN OF VERNIERS.	ANGLE. MEAN D AND R.	REMARKS.
	A. M.			° ′	″	″	″	″	° ′ ″	
Alpha.......	7.30	D	0	00 00	10	05	00	05·0		Weather, fair,
Beta.........			1	35 10						calm: at-
		D	6	211 01	00	05	$\overline{55}$	00·0	35 10 09·2	mosphere,
		R	6	62 01	40	50	55	48·3	08·0	unsteady.
Beta.........	7.38	R	6	211 01	10	20	05	11·7	06·1	
Alpha.......		D	6	00 00	20	20	10	16·7	09·2	
Beta.........	7.44	D	1	45 18						
Gamma......		D	6	271 49	05	10	00	05·0	45 18 08·0	
		R	6	183 38	05	$\overline{55}$	00	00·0	09·2	
Gamma......	7.52	R	6	271 49	20	25	05	16·7	07·2	
Beta.........		D	6	00 00	40	45	30	38·3	06·4	
Alpha.......	8.00	D	1	80 29						
Gamma......		D	6	122 50	00	05	$\overline{55}$	00·0	80 28 13·6	
		R	6	245 39	20	30	25	25·0	14·2	
Gamma......	8.06	R	6	122 50	10	20	05	11·7	12·2	
Alpha.......		D	6	00 00	55	55	60	56·7	12·5	

RECORD FOR HORIZONTAL DIRECTIONS.

Station, Uncompahgre. State, Colorado. County, Hinsdale. Date, Aug. 20, 1895. Observer, William Eimbeck.
Instrument, 20-in. Theodolite, No. 5. Position, I.

SERIES AND NO.	OBJECTS OBSERVED.	TIME.	TEL. D OR R.	MIC.	°	′	d. BACKWARD. b.	d. FORWARD. f.	$b-f=r$.	MEAN d. $\frac{b+f}{2}$	CORR'N FOR RUN c.	CORRECTED MEAN. ″
1	Mark.	6.22 P. M.	D	A	00	00	10·7	09·3				
				B			05·7	05·3				
				C			07·0	08·1				
							07·8	07·6	+0·2	07·7	+0·2	07·9
	Mesa.		D	A	71	50	18·0	18·0				
				B			21·0	23·3				
				C			22·3	21·8				
							20·4	21·0	−0·6	20·7	+0·1	20·8
	Treasury.		D	A	122	33	52·3	52·2				
				B			63·4	63·5				
				C			65·2	63·3				
							60·3	59·7	+0·6	60·0	−0·1	59·9
	Elbert.		D	A	142	52	02·2	02·5				
				B			19·8	20·2				
				C			11·1	09·7				
							11·0	10·8	+0·2	10·9	0·0	10·9
	Elbert.	6.40 P. M.	R	A	322	52	18·2	18·0				
				B			18·5	17·7				
				C			12·5	14·7				
							16·4	16·8	−0·4	16·6	0·0	16·6
	Treasury.		R	A	302	34	07·0	05·2				
				B			10·7	10·1				
				C			01·0	02·0				
							06·2	05·8	+0·4	06·0	−0·1	05·9

REMARKS.

Weather, partly cloudy. Wind, light, W.
Barometer, 0·4569^{m}.
Attached ther. +6·8° Cent.

Run obtained from 231 observations, $\Sigma r = +79·3^{d}$.
Mean run $= +0·342^{d}$.

Table for Run Corr'n (c).

m	c	m
′ d	d	′ d
0 00·0		5 00·0
	+0·2—	
0 18·3		4 41·7
	+0·1—	
1 46·1		3 13·9
	+0·0—	
2 30·0		2 30·0

ABSTRACT OF HORIZONTAL ANGLES.

Station, Omega. Observer, O. K. Repeating Theodolite, No. 52.

DATE.	HOUR.	STATIONS OBSERVED.	TEL. D OR R.	NO. OF REP'S.	ANGLE.	MEAN OF D AND R.	STATION ADJ'MT CORR.	MEAN ANGLE.
1895.	A. M.				° ′ ″	″	″	° ′ ″
June 7.	7.30	Alpha	D	6	35 10 09·2			
		Beta	R	6	08·0			
	7.38		R	6	06·1			
			D	6	09·2	08·1	−0·9	35 10 07·2
	7.44	Beta	D	6	45 18 08·0			
		Gamma	R	6	09·2			
	7.52		R	6	07·2			
			D	6	06·4	07·7	−0·9	45 18 06·8
	8.00	Alpha	D	6	80 28 13·6			
		Gamma	R	6	14·2			
	8.06		R	6	12·2			
			D	6	12·5	13·1	+0·9	80 28 14·0

ABSTRACT OF HORIZONTAL DIRECTIONS.

Station, Uncompahgre. Observer, William Eimbeck. Instrument, 20-in. Theod., No. 5.

DATE, POSITION, SERIES.	MARK.	MESA.	TREASURY.	ELBERT.	EXPLANATION.
August 20, 1895	00° 00′	71° 50′	122° 33′	142° 52′	
P. M.	07·9″	20·8″	59·9″	10·9″	Tel. direct.
	10·2	26·0	65·9	16·6	Tel. reversed.
Position I	09·0	23·4	62·9	13·8	Mean.
Series 1	00·0	14·4	53·9	4·8	Referred to mark.
August 24	18·6	33·2	13·0	26·2	Tel. D.
A. M.	23·1	39·7	18·1	30·8	Tel. R.
	20·8	36·4	15·6	28·5	Mean.
Position V	00·0	15·6	54·8	7·7	Ref. to mark.
Series 2					

BASE-LINES.

606. Site of Base-Line. Since the length of every line in a system of triangulation depends directly upon the base-lines of that system, it is necessary to consider everything that tends to increase the accuracy of the determination of the lengths of the base-lines.

The triangulation is usually carried across country over which it is impracticable to measure directly, hence the base-line is located at some place favorable for a short connection with the main scheme, and also for easy and accurate determination of its length.

The site for the base-line should be carefully investigated, and

that position selected, other things being equal, where the soil is the most stable and the surface most level, over which the measurement is to be made. The ends should be intervisible in order to benefit by the whole length.

607. Preparation of the Line. This must necessarily vary with the apparatus used, but should be sufficient to permit quick work with the apparatus. If bars are used, the surface should be no more disturbed than to permit the prompt placing of the tripod supports.

The ends of the line must be marked very carefully, and in a permanent manner. On the United States Coast and Geodetic Survey the following method is usually employed: A stone post about 2 feet by 6 inches by 6 inches is sunk so that its upper face is about 3 feet below the surface of the ground. In it is set a copper bolt with fine cross lines on its surface (or a needle hole) to mark the exact point. In soil at all unstable a layer of concrete is placed above and around this post, leaving only a small space directly over the bolt to enable easy reference to the underground mark. On the concrete foundation is placed the block of stone that carries the surface mark, a bolt similar to the underground bolt, or one terminating in a spherical head, the latter being preferred where the optical apparatus and Repsold cut-off are used. Witness or reference marks should be placed near the end marks, and their positions with reference to the station marks very carefully determined and noted. These reference marks should be given as little prominence as possible, so that they might escape notice in case the station mark is disturbed, and thus make it possible to reset the latter.

A full description should be filed with the records, together with a sketch showing the location, and also a general description of the whole locality.

Intermediate stones should be set on the line at about half-mile or kilometre intervals. A good telescope should be used in this work, set either over one end and pointed to the other, or set in the middle by trial, so that the telescope in transiting, both direct and reversed, will point to either end mark.

608. Base Apparatus. Numerous forms of apparatus have been used in the measurement of geodetic base-lines. Rods of wood, glass, metals, metallic tapes, and chains have all been under trial. Wood was found to possess properties the effects of which could not be wholly eliminated, such as changing its volume by the absorption of moisture; glass and zinc were found to be alike in that their volume change is not wholly coincident with their temperature change, being rather a function of the temperature changes for an indefinite previous period. Hence these materials have been discarded in the construction of base apparatus.

The essential features sought for in a base-measuring apparatus are:

1. The terminal points, used as measuring extremities, must during the operation remain at an unvarying distance apart, or the variations therefrom must admit of easy and accurate determination.

2. The distance between these extremities must be compared with a standard unit to the utmost degree of accuracy, and the absolute length determined.

3. In its construction provision must be made to secure readiness in transportation, ease and rapidity in handling, stability in supports, and accuracy in ascertaining exact contact and deviation from a horizontal line.

The various forms of base apparatus may be classified as follows:

1. Contact apparatus, including compensating bimetallic, monometallic bars.

2. Optical apparatus, embracing compensating bimetallic, monometallic, bars in ice.

3. Tapes and wires.

609. By **Contact Apparatus** is meant those forms of apparatus in which the ends of the bars are brought into successive contacts, or the length of the bar is the distance between its extreme ends. Contact bars are usually fitted with a slide and spring at the rear end for use in making a fine contact. The length of the bar, then, is the distance between its end points when a line on the slide coincides with a line on the tube surrounding the slide.

610. In **Optical Apparatus** the measure is fixed by fine lines near the ends of the bar. The bar is placed in the base-line to be measured and microscopes set so as to point to the marks on the ends of the bar. The forward microscope is then left to mark the forward end of the bar until the rear end has been adjusted under it—i. e., the end of the measure is marked by a microscope which is considered immovable while the bar is moved forward until the rear end is in the place the forward end has just vacated.

611. A **Compensating Apparatus** is intended to be so constructed that its ends are always at the same distance apart, irrespective of any changes of temperature. Several different forms of compensating apparatus have been constructed and used, but they are all unsatisfactory, owing to mechanical difficulties in the construction.

The Colby apparatus used in India has given results fairly good, as also the Bache-Würdemann of the United States Coast and Geodetic Survey, but the best compensating apparatus thus far constructed is the Schott apparatus used to measure two bases in California.* As the compensating element is composed of zinc, the apparatus is not as satisfactory as is desirable.

612. References. As a full description of the various forms of base-measuring apparatus would be out of place in a work of this kind, only those will be discussed that are in use, or are likely to be used, in this country.

For the benefit of any one wishing to enter more fully into the study of base apparatus, the following references are appended:

For Col. Ibanez apparatus, used in Spain (monometallic, optical), see "Memorias del Instituto Geographico," vol. iii, 1881, and vol. v, 1884.

Colby apparatus (compensating, optical, bimetallic), "Report of Great Trigonometrical Survey of India," vol. i. Clarke, "Geodesy," p. 163.

Bessel (bimetallic, non-compensating, contact), "Gradmessung in Ostpreussen," pp. 1–58.

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* See Art. 612.

Brunner (bimetallic, non-compensating, optical), "Generalbericht der Europ. Gradmessung," 1878, 1879.

Bache-Würdemann (contact, compensating, bimetallic), "United States Coast and Geodetic Survey Report," 1854.

Repsold (optical, bimetallic, non-compensating), "Report of Triangulation of the Great Lakes," "Professional Papers, Corps of Engineers, U. S. A.," No. 24.

Schott (compensating, bimetallic, contact slide), "United States Coast and Geodetic Survey Report," 1882.

Iced bar apparatus, 100-metre steel tapes, and secondary apparatus, "United States Coast and Geodetic Survey Report," 1892.

613. Secondary Base Apparatus. This apparatus as used on the United States Coast and Geodetic Survey consists of two cylindrical steel rods, each 5 metres long and 9 millimetres in diameter, with the usual contact slides. The slide end of each rod carries a white agate knife edge 3 millimetres wide, the other a circular plane surface about 3 millimetres in diameter. The index and slide piece of each rod have each three lines ruled on them. When the middle lines of the slide and index are in coincidence the distance between the agate plane at one end and the knife edge at the other is the length of the rod.

Each steel rod is encased in a built-up wooden bar and rests directly in a longitudinal groove in its middle. The wooden bars are 7·5 centimetres wide, 14 centimetres thick, and 4·9 metres long, so that about 10 centimetres of each rod is exposed to the air.

Each rod carries two mercurial thermometers. They are inserted in opposite sides of the bar and are in metallic contact with the rod and not the wood. Each is one metre from the end of the bar.

The wooden bar is in turn surrounded with a white canvas cover lined with half an inch of cotton.

614. Trestles. These rods are placed upon trestles made by screwing a frame with vertical motion and levels on the head of a tripod with double legs of the usual construction. The rear tripod has a platform for the support of the bar, and the forward tripod a roller to allow a longitudinal motion by merely lifting the

rear end when making the contact. The platform has a vertical motion by means of a wooden wedge and springs; clamps and guide wheels are introduced whenever necessary to secure greatest efficiency.

615. Sectors. The inclinations of the rods are obtained by means of sectors attached to the side of each rod. Each sector consists of an arc of a circle, and carries two verniers 180° apart, reading to 10′. In order to adjust the sectors—i. e., make the reading 0° when the rod is horizontal—a theodolite or a leveling instrument is used to bring the contact ends into a horizontal plane. When in this position the level bubble of the sector is brought to the center, the verniers having been set at 0°, or the vernier reading is taken when the bubble is in the center and this value applied to each reading as an index correction.

In general, the sector errors should be determined at least three times a day.

Each sector should be protected against the sun by a hood made of canvas stretched on a wire frame attached to the bar.

616. Alinement. The bars are alined during measurement by bringing the rear end of the rod over the starting point or into contact with the forward agate of the rear bar, and keeping it there while the forward end of the bar is alined by pointing a small telescope on a signal usually set on the next section stone ahead. A small alining telescope is mounted on each bar 25 centimetres from the end. The base of the uprights carrying the telescope has a sliding motion on an arc concentric with the rear end of the bar, and is provided with adjusting screws so that the line of collimation of the telescope may be made parallel with the rod. It also carries a level, by means of which it can be made horizontal and clamped in the vertical plane passing longitudinally through the rod.

The ends of this form of apparatus are referred to the ground by setting a theodolite or transit sector at a short distance from the bar and on a line perpendicular to it. At the start the slide is clamped in coincidence with the index and the knife edge brought over the starting point by means of the transit mounted at right

angles to the base-line and just far enough away to give a good definition of the objects, as seen through the telescope, without changing its focus. The transit should be carefully leveled and used both direct and reversed, to eliminate any inclination of the axis of the telescope.

Whenever the work is interrupted the end of the bar in place should be transferred to a ground mark (as a cross on a copper tack driven into the top of a stake) by means of the transit sector, thus eliminating any change that might occur in the apparatus while standing. If, upon resuming work, the end of the rod is adjusted over this mark, no corrections of such a nature need be considered.

617. The Holton Base. This secondary apparatus was used to measure the Holton base* in Indiana in 1891. Two measurements of the whole base were made, and also two extra measurements of the standard kilometre. The resulting length is 5500·816 metres ± 3·2 millimetres. This probable error includes all the errors known necessary to refer the length of the base to the international metre.

The following conclusions were derived from this work, particularly from the measurements of the standard kilometre:

1. That a speed of 1·5 kilometre per day of eight hours may be attained and maintained without detriment to accuracy.

2. That if the temperature of the bars were known the accuracy of the measurement would leave nothing to be desired, the probable error of a single measurement being about one part in one million.

3. That the principal source of error is the uncertainty between the indicated and true temperature of the bars.

618. Steel Tapes. The long steel tape as a base-measuring apparatus is gradually gaining in the confidence of geodesists, and will probably soon supersede all other forms of base apparatus excepting when the highest possible degree of accuracy is required.

One of the first to experiment with long steel tapes for the meas-

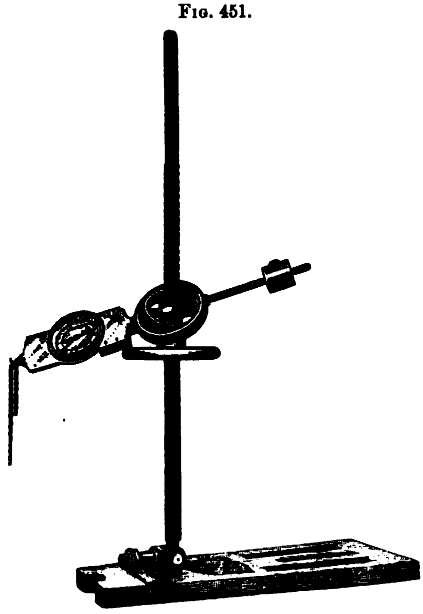
* "Report of the United States Coast and Geodetic Survey," 1892, App. 8.

urement of geodetic base-lines was Prof. Edward Jaderin, of Stockholm, Sweden.* The Mississippi and Missouri River Commissions have also investigated and used such tapes.

On the United States Coast and Geodetic Survey investigations of the practicability of the measurement of base-lines of precision with long steel tapes was begun in 1890 by Assistant R. S. Woodward.

The tapes, as finally devised by him, are of steel, 101·01 metres long, 6·34 millimetres by 0·47 millimetre in cross-section, and weigh 22·3 grammes per metre of length. They are subdivided into 20-metre spaces by graduations on the surface of the tape itself. The end graduations fall about one half metre from the tape ends, which terminate in loops formed by annealing and riveting the tape back on itself. The surface is of a dull-black color wherever not polished to receive the graduation marks.

FIG. 451.



619. Tape Stretchers. The accompanying cut (Fig. 451) shows the apparatus devised to give tension and alinement to the tape. It consists of a lever hinged by a universal joint to a platform on which the operator stands. This lever is made of a piece of steel tubing terminating in a hickory handle. Along the upper two thirds of the tube is cut a screw thread on which a wheel nut plays freely. This nut gives a vertical motion to a gimbal-jointed support to which a spring balance is attached. Thus all the motions are obtained which are necessary to aline the tape and give it the proper tension.

* "Zeitschr. f. Instrumentkunde," 1885, pp. 362, etc., and "United States Coast and Geodetic Survey Report," 1893, App. 5.

The balance should be one from which ounces can be read, as the operator can easily hold the tension within an ounce. In order to prevent overstraining of the tape by accident, a breaking link is inserted between the end of the tape and the balance. It is so constructed that it parts whenever the tension becomes greater than about 14 kilogrammes, or 30 pounds.

620. Thermometers. Mercurial thermometers graduated to read half degrees, or so that the nearest fifth of a degree can be obtained with certainty, were used. To determine the temperature of the tape, light wire hoops were attached to the upper ends of the stems, so that the thermometer could be suspended by a cord and isolated from adjacent masses, or whirled in the air when necessary.

At least two thermometers should be used for a tape length. They should be distributed proportionately along the tape. For the most precise work the thermometers should be read both before and after the measure, the mean being taken as the temperature.

621. Supporting and Marking Position of Tape. When in use for measuring a line the tape is supported at equal intervals of 10 metres, or 20 metres, throughout its length. The most convenient supports, and amply sufficient, are steel wire nails driven into stakes set at the proper interval along the line. The nails are ranged into a straight line by means of a theodolite. The stakes for the tape ends are more solid than the others, and have small tables on their tops, to which are nailed plates of zinc, 5 centimetres by 20 centimetres, the longer sides being parallel with the line.

The position of the forward-end graduation of the tape is marked on the plates as the work progresses by means of a sharp bradawl held against the edge of a try-square, alined against the edge of the plate. The rear-end graduation is brought in coincidence with these marks, excepting where the expansion or contraction is so great that the forward end falls too near the edge of the plate, in which case a "set-up" or "set-back" is introduced—i. e., the rear end is held over a mark placed at a known distance back or forward from the one already marked. These plates, if numbered,

oriented, and dated, can be filed as part of the record, and are particularly valuable whenever several measures are made.

622. Balances. As the spring balance used to give tension to the tape is adjusted to read correct tension when in a vertical position, it will indicate (by its face reading) less than the actual tension when in a horizontal position. An appreciable index correction is also liable from wear and other causes. To determine the actual tension in any case let r be the index correction. It is the reading of the index when the balance is vertical, hook end down, and without load on the hook; it is minus when index reads greater than zero, and plus when it reads less. Let R be the reading of the index when the balance is suspended by its hook, hook end up. Let W be the total weight of the balance found by weighing it on another balance. Then, if T denote the observed or face reading of the balance when horizontal, and T_1 the corresponding actual tension,

$$T = T_1 + r + \tfrac{1}{2}(W - R - r) = T_1 + \tfrac{1}{2}(W - R + r). \quad [1.]$$

The standard tension adopted in all the base measures was 25 pounds 9 ounces, which corresponded very closely with an indicated tension of 25 pounds for the balances used.

623. Grade Corrections. The difference in height between successive marking stakes or tables having been obtained by leveling, the correction to the inclination measure, in order to get the horizontal distance, is easily obtained.

Let h = the difference in altitude of the two ends of the tape when in place over two successive marking tables.

Let s = the distance between the ends of the tape when in place over the tables. If c = grade correction, then $c + s = \sqrt{s^2 - h^2}$.

Whence
$$c = -\frac{h^2}{2s} - \frac{h^4}{8s^3} - \ldots\ldots \quad [2.]$$

Since s is large as compared with h, this series converges very rapidly. For a 100-metre tape,

$$\frac{h^4}{8s^3} < 0{\cdot}01 \text{ millimetre for } h < 2{\cdot}9 \text{ metres.}$$

As values of h as great as 3 metres are rarely met, we may for all ordinary cases use (for a 100-metre tape),

$c = -5\ h^2$, c being in millimetres if h is taken in metres.

624. Computation Formulas.

Let T = actual or working length of tape;

$A + a$ = standard length, A being in round numbers and a a small excess for convenience of computation;

$\propto$ = coefficient of expansion for the whole tape length, per degree;

t = temperature of tape referred to its standard temperature, after applying the index corrections of the thermometers;

Then $$T = A + a + \propto t.$$

N = number of tape lengths used;

C = correction for fraction of tape lengths or end corrections;

D = algebraic sum of set-ups and set-backs, the latter having a minus sign;

G = sum of grade corrections, always negative;

Then the total length of the base-line is given by:

$$\Sigma T = NA + Na + \propto \Sigma t + C + D + G. \qquad [3.]$$

When the tension applied to a tape differs from the normal, or tension under which the tape was standardized, a change in length results. Likewise, a change in length will occur when the interval between equidistant supports is changed, or when some of the supports are omitted. It is often convenient to omit one or more supports, as in crossing a ravine, road, etc. Not infrequently, also, tapes are standardized by laying them on a flat or mural standard, and it then becomes essential to show the shortening they undergo when supported at finite intervals, instead of throughout their length.

The following formulas apply to these cases:

Let w = weight of tape per unit of length;

τ = the applied tension;

$b = \dfrac{w}{\tau}$;

n = number of sections into which the tape is divided by equidistant supports;

l = length of any such section;

L = normal length of tape, or right-line distance between its end marks when under standard tension;

$L = \Sigma l = nl$, approximately;

μ = reciprocal of product of the modulus of elasticity of the tape by the area of its cross-section.

1. The change in length (ΔL_1) in L due to a change, $\Delta \tau$, in the tension, is:

$$\Delta L_1 = nl\mu \Delta \tau + \tfrac{1}{12} b^2 n l^3 \frac{\Delta \tau}{\tau}. \qquad [4.]$$

Example: United States Coast and Geodetic Survey Tape No. 85, 100 metres.

$n = 10$, $l = 10$ metres, $w = 22{\cdot}32$ grammes per metre,

$\tau = 25{\cdot}5$ pounds $= 11566{\cdot}66$ grammes.

$b^2 = 372 \times 10^{-8}$, $\mu = 16 \times 10^{-9}$ for gramme as unit,

$\mu = 450 \times 10^{-9}$ for ounce as unit.

(Cross-section of tape is 6·34 millimetres $\times$ 0·47 millimetre, or 0·0298 square centimetre, and modulus of elasticity, $2{\cdot}1 \times 10^6$ kilogrammes per square centimetre, or 30×10^6 pounds per square inch.)

Hence, for $\Delta \tau = 1$ ounce,

$$nl\mu\ \Delta \tau = 10 \times 10 \text{ metres} \times 450 \times 10^{-9} = 0{\cdot}0450 \text{ millimetre}$$

$$\tfrac{1}{12} b^2 \mu l^3 \frac{\Delta \tau}{\tau} = \tfrac{1}{12} \times 372 \times 10^{-8} \times 10^7 \times \tfrac{1}{408} = 0{\cdot}0076 \quad \text{"}$$

$$\Delta L_1 = \qquad 0{\cdot}0526 \quad \text{"}$$

It may be observed that ΔL_1 can be measured directly by increasing and decreasing the tension in the vicinity of the assumed value.

2. Suppose a given tape to be divided by its equidistant supports—first, into n_1 sections of length l_1; and, second, into n_2 sections of length l_2. Then assuming the tension the same in both cases, if n_2 is greater than n_1, the difference in distance between the end graduations of the tape, in the two cases, say ΔL_2, will be expressed by,

$$\Delta L_2 = \Sigma(l_2 - l_1) = \tfrac{1}{24} b^2\, (n_1\, l_1^3 - n_2\, l_2^3). \qquad [5.]$$

Example: $n_1 = 5$; $n_2 = 10$; $l_1 = 20$; $l_2 = 10$, for 100-metre tape No. 85, given in last example,

$b^2 = 372 \times 10^{-8}$, as above,

Hence,

$$\triangle L_2 = \tfrac{1}{24} \times 372 \times 10^{-8} [5 (20)^3 - 10 (10)^3] = 4{\cdot}65 \text{ millimetres.}$$

If a single support is omitted we have only to make $n_2 = 2$ and $n_1 = 1$, and $l_1 = 2l_2$ in equation (5).

If m consecutive supports are omitted the tape is shortened by

$$\tfrac{1}{24} m (m + 1) (m + 2) b^2 l^3, \qquad [6.]$$

l being the length of a section where no supports are omitted.

3. When a tape is supported throughout its length, as when lying on a horizontal plane, or mural standard, n_2 in formula (5) becomes infinite, and l_2 infinitesimal.

If we call L_0 the length of the tape in this case, and $\triangle L_3$ the amount it is shortened by supporting it at equidistant intervals, n in number, and of length l, then

$$\triangle L_3 = L_0 - \Sigma l = \frac{n}{24} b^2 l^3. \qquad [7.]$$

4. Lastly, it will be of interest to have an expression for the change in length of a tape due to a change in its weight per unit of length. Such a change may occur by reason of a deposit of dew on the tape, or from the wear of the same.

Let $\triangle w =$ change in weight per unit length, or the increase to w, and let the corresponding change in the total length of the tape be denoted by $\triangle L_4$.

Then, assuming as before that the number of sections in the tape is n, and that their lengths are each l approximately,

$$\triangle L_4 = - \tfrac{1}{12} n b^2 l^3 \frac{\Delta w}{w}. \qquad [8.]$$

This formula shows, for example, that when tape No. 85 is supported at equidistant intervals of 10 metres, and is under a tension of 25·5 pounds, or when $l = 10$ metres, $n = 10$, $b^2 = 372 \times 10^{-8}$ we have

$$\triangle L_4 = - 3{\cdot}1 \text{ millimetres} \times \frac{\Delta w}{w}.$$

Thus, in order to produce a change of one millionth part in the length of the tape, or in order that $\triangle L_4$ may be 0·1 millimetre, we must have $\triangle w = \dfrac{w}{31}$.

In order to produce the best results, the tape should be standardized under exactly similar conditions to those that exist in the

field where the base is to be measured; equal intervals between supports, same tension as shown by same balances, and at temperatures as near those of the field as possible. As the thermometers do not change in temperature at the same rate as the tapes, this source of error should be eliminated as much as possible, either by making one measurement under falling temperatures and another while the temperature is rising, or by making half the measurement with rising and half with falling temperatures.

As the work of measuring a base with the tape is comparatively small after the line is prepared, it is recommended that at least two measures be made, and preferably four or more, using two different tapes.

The coefficient of expansion must always be obtained for the apparatus itself, as the same material often has different degrees of expansion.

Assistant Woodward measured two bases with the 100-metre steel tapes, one 5·5 kilometres, and the other 3·9 kilometres in length. The first was the Holton base in Indiana, where the iced-bar and secondary apparatus of the United States Coast and Geodetic Survey were also used.

The tapes were standardized by means of the iced-bar apparatus, both by comparison with the kilometre of the base measured directly with the iced bar, and also by stretching over a hundred-metre comparator under a shed, also measured with the iced bar.

This base was measured in sections, the number of measures of each section varying from four to twenty-one; five, however, was the average, leaving out the section measured with the iced bar, that being the one where twenty-one measures were made.

Measurements were made with varying conditions as to cloudiness and sunshine, wind and rain, and also at night. The results showed that a very good determination of the length of a base could be obtained if the tapes were standardized under exactly similar conditions to those during the measurement. But the best results were obtained from night measures, where the tapes were standardized under a shed open only to the north.

By using the night measures of the Holton base only, the prob-

able error of the whole length of 5500·832 metres was found to be $\pm$ 3·68 millimetres, or $\frac{1}{1,500,000}$ of the length.

The other base, 3870·512 metres in length, was obtained with a probable error of $\pm$ 3·1 millimetres, or $\frac{1}{1,260,000}$ part of its length. Hence we may conclude that the length of a base-line can be obtained by means of long steel tapes with a relative error of less than $\frac{1}{1,000,000}$ of its length by measuring under favorable conditions.

For the mathematical theory of steel tapes, see "Report of the Coast and Geodetic Survey," 1892, pages 480–489.

625. Iced-Bar Apparatus. The determination of the actual temperature of the measuring apparatus has always been the greatest difficulty in the determination of the length of a base. The use of melting ice in thermometry naturally suggests its adaptability for keeping a measuring bar at a fixed or standard length during a measurement.

Although outlines of schemes for such an apparatus have been made by several,* the first to complete such a scheme was R. S. Woodward, while assistant on the Coast and Geodetic Survey.

As this apparatus, or some modification of it, will without doubt be used wherever very accurate measurements are desired, it is presented here as fully as a work of this character admits.

The iced-bar apparatus is optical and monometallic. It consists of a rigid bar of tire steel rectangular in shape. It is 5·02 metres long, 8 millimetres thick, and 32 millimetres deep. The upper half is cut away about 2 centimetres at either end to receive the graduation plugs of platinum iridium, which are inserted so that their upper surfaces lie in the neutral surface of the bar, thus making them at a constant distance apart under all conditions of flexure to which the bar may be subjected. Three lines are ruled on the surface of these plugs, two in a direction longitudinal and one transverse to the length of the bar. The longitudinal ones, serving merely to limit the part of the transverse line used, are 0·2 millimetres apart.

This bar is known as No. 17, or B. 17. It is carried in a Y-shaped

* See Wright's "Adjustment of Observations," p. 360.

trough in which ice is placed. This trough is mounted on two cars which move on tracks either stationary or portable. The ends of the bar were marked by means of micrometer microscopes mounted on posts previously ranged into line and set firmly in the ground. The microscopes are carried forward as the measure of the line progresses, and the bar in the ice likewise easily rolled forward on the cars along the track.

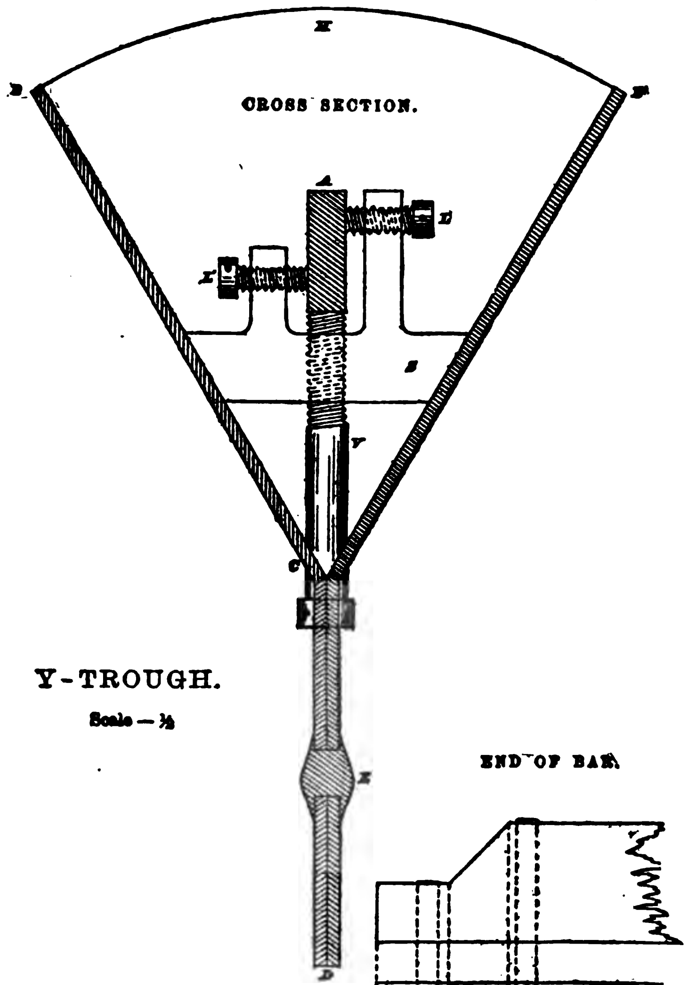
626. Alinement. To secure the alinement of the bar, eleven German-silver plugs 5 millimetres in diameter are inserted at intervals of 495 millimetres along the bar and project about 1 millimetre above its top surface. The upper surfaces of these plugs are all the same distance from the neutral surface of the bar within a few hundredths of a millimetre. On the top of each a fine line is ruled in the direction of the bar. The length of the bar, as regards alinement, is defined to be the distance between the transverse graduation marks when the upper surfaces of the alinement plugs are all in the same plane, and when the lines on these plugs are in one straight line.

The former is tested by placing a striding level with its feet resting on alternate plugs, and the latter by stretching a fine wire or string from one end of the bar to the other.

627. Y-Trough. An important and distinct feature of this apparatus is the Y-trough which supports the bar, keeps it alined, and carries the ice load necessary to control the bar's temperature. The accompanying sketch (Fig. 452) shows a cross-section of the bar and trough. The trough is made of two steel plates 5·14 metres long and 0·255 metre by 3 millimetres, bent at an angle A B C and riveted as shown at E. The bar, shown in cross-section at D, is supported every half metre of its length by saddles, one of which is shown in the figure S; each saddle is firmly fastened to the sides of the trough, and carries one vertical and two horizontal adjusting screws, shown at V L L′. At the end saddles the lateral adjusting screws are both at the same elevation as L′ the lower, and the intermediate ones are alternately high and low to prevent the possibility of pinching the bar, and also to afford a means of rotating

the bar slightly about its longitudinal axis, so that the bar can be made vertical independent of the trough. The trough is very rigid

Fig. 452.



with respect to vertical stresses, and weighs about 82 kilogrammes exclusive of bar and ice load. The ends are fitted with V-shaped wooden blocks to prevent the ice falling out.

The whole trough is covered by a closely fitting jacket of heavy

white cotton felt, which protects it and its load from direct radiation.

Grades are marked by means of a very rigid sector, reading by two opposite verniers to 10″, attached to the middle of the trough.

When the apparatus is in use the Y-trough is filled with pulverized ice. By means of its weight and the shape of the trough the ice is kept in close contact with the bar, and the jarring given the apparatus while moving from one position to the next is sufficient to overcome any tendency to pack.

The Y-trough is mounted on two cars, the bolsters of which are attached to the trough 40 centimetres from either end. Each bolster is rigidly attached to the trough above, and to a jackscrew below, for raising or lowering the trough. The jackscrew is attached to a slide rest connected rigidly with the base of the car. The slide rests are provided with slow-motion screws to enable the observer to adjust the bar both along the line and transverse to it.

The cars have each three wheels, and run on a portable track 30 centimetres wide and in sections five metres in length. Thus the apparatus is made to rest on but one section when in use.

The micrometer microscopes used with this apparatus are similar to those used with the Repsold apparatus on the survey of the Great Lakes.* They are provided with levels and leveling screws, so that their axes may be made vertical, and are mounted on slide rests which allow a motion of two centimetres in the direction of the line and also transverse to it. The micrometer screws have a value of 0·1 millimetre per turn, hence each division of the head is 0·001 millimetre; and as tenths of a division can be easily estimated, the smallest unit considered is a tenth of a micron, a micron being the millionth part of a metre.

628. Cut-off Cylinder. The method of referring the end of the bar (i. e., the line marking the end of the measure) to the surface marks is essentially the same as that employed with the Repsold apparatus. The surface mark is the fiducial point of a spherical-

* "Professional Papers," Corps of Engineers, U. S. A., No. 24.

headed bolt imbedded in a stone set firmly in the ground. To refer to this the cut-off cylinder is used. It consists of a cylinder terminating at the lower end in a conical hole that fits over the spherical bolt head. The upper end is provided with a level and scale which are placed parallel with the line measured. The scale is brought into the focus of the microscope, whose position with reference to the bolt head is desired. A rack and pinion motion accomplishes the latter adjustment. When thus placed readings are made on the scale and the position of the level bubble noted. The cylinder is then turned 180° in azimuth and the scale and level again observed. From these observations, and the height of the scale above the bolt, the horizontal distance in the direction of the line between the microscope zero and the fiducial point of the bolt head may be accurately determined.

629. Method of Measurement. The position of the microscope having been determined by the cut-off cylinder, with respect to the surface mark, as explained above, the rear end of the bar is brought to focus under that microscope by the rear-end observer. By means of the lever which grips the track and hinges on the car the latter observer holds the bar near to bisection under his microscope, while the front-end observer brings his microscope into position over the front end of the bar by moving the microscope, the trough, or both. When the bar is adjusted at both ends the rear-end observer brings the rear-end graduation accurately to center of his micrometer wires by means of the lever without moving the micrometer screw. Simultaneously the front observer brings his micrometer wires to bisect the front-end graduation mark by moving the microscope, turning the micrometer screw, or both. Then each observer reads his micrometer and observers exchange places. The rear observer carries his lever with him and applies it to the front car, and each observer repeats his former operation—i. e., the rear observer brings the bar into coincidence with his micrometer, whether he is at the front end or at the rear, and the front-end observer always moves his micrometer. This process eliminates the personal equation of the observers and checks any blunders of whole revolutions or the like in reading the microscopes, each being read four times, and the

four readings being the same within a few microns. The probable error of a bisection has been found to be less than one micron ($\pm 1^{\mu}$).

When the bar is in position the third observer measures the distance of the front end (and rear end at starting) of the axis of the bar from the reference line, and adjusts the sector-level bubble to center, taking care all the time to keep away from the microscope posts when the bar is observed. The grade-sector is then read and the bar rolled rapidly forward to a new position. The observers stand on platforms which do not touch the ground within a metre of the posts bearing the microscopes. The speed of measurement varies somewhat with circumstances; 100 metres per hour can be easily maintained after a little practice.

630. Computation Formulas. Let R and L be the means of the reading of the microscopes on the right- and left-hand ends of the bar respectively, the correction then for micrometer readings (supposing them to increase from right to left) is $\Sigma\,(L - R)$. If R_c and L_c are the mean micrometer readings on the cut-off scale at the right- and left-hand ends of the line respectively, the correction is $-(L_c - R_c)$. If there are several surface marks in a line each section may be considered separately and the total correction found by the sum of the different sections, or $-\Sigma\,(L_c - R_c)$.

Let S_r and S_l represent the reading on the cut-off scales at the two ends respectively—i. e., the distance from the zero of the scale to the mark observed upon by the microscope, the zero being at the center of the vertical cylinder. Then the correction is $\Sigma\,(S_l - S_r)$ when S_l and S_r are plus or minus, according as the image of the graduation numbers appears inverted or erect.

Let I_l and I_r indicate the inclinations of the cut-off cylinders at the two ends of the section, and H_l and H_r the corresponding heights of the cut-off scales above the surface marks. Since the inclinations are small the correction can be represented by $\Sigma\,(H_l\,I_l - H_r\,I_r)$ where the inclinations are expressed in arc and both are supposed to be toward the right from the vertical. If l_1, l_2, r_1, r_2 are the left- and right-hand readings of the level bubble, and v the value of one division of the level in seconds of

arc, $I = \frac{1}{4}(l_1 + l_2 - r_1 - r_2)\frac{v}{\rho}$ where $\rho = 206264{\cdot}8''$ or number of seconds in the radius.

The correction for grade or slope of any bar is given by the formula $-\frac{(\Delta h)^2}{2s}$, Δh being the difference in height of the two ends of the bar and s the length of the bar. For Δh in millimetres this gives a correction of $-0{\cdot}1(\Delta h)$ microns per bar length, since $s = 5$ metres very nearly. The same formula gives the correction for alinement. If the grades are observed with a sector it is more convenient to tabulate the corrections for different angles for the length of the bar and thus obtain them directly from the table.

If now we have laid N bars in measuring a distance D between two ends of a base-line, the formula for getting D may be written thus, collecting all those just described:

$$D = NB_{17} + \Sigma(L - R) - \Sigma'(L_c - R_c) + \Sigma'(S_l - S_r) + \Sigma'(H_l I_l - H_r I_r) - \Sigma g - \Sigma a, \ldots .$$

where g is the grade correction for each bar length and a the correction for alinement. Σ means the summation of the quantities obtained at each individual position of the bar, and Σ' those at the sections where cut-offs were introduced. If now the length of the bar used (B_{17}) be known, the length of the whole base is readily computed from this formula.

631. Standardization of B_{17}. The determination of the length of B_{17} in terms of the international prototype metre was the most important and difficult operation attending the use of this apparatus. The general plan followed was to make a careful comparison of the bar and the prototype metre with both packed in melting ice, thus dispensing entirely with the use of thermometers.

The first comparison was made in the comparing room at the Coast and Geodetic Survey Office at Washington, D. C. A large iron I beam supported at its ends on brick piers served as the comparator—i. e., the microscopes were fastened to it, and the objects to be compared brought into their foci. The distance of five metres was measured with the prototype and then the bar B_{17} compared, the manner of using the prototype being similar to the use of B_{17} in the field.

The results were not as good as was desired, the microscopes changing their relative positions seemingly; hence a bar similar to B_{17} was constructed and plugs inserted at each metre in its length for direct comparison with the prototype, and afterward a direct comparison of B_{17} with this bar. These results also proved unsatisfactory, and it was finally decided that a comparator must be constructed in a more stable manner. With this object in view an apparatus was constructed in the grounds of the Coast and Geodetic Survey. It consisted of six brick piers resting on a continuous foundation of concrete, all set in Portland cement, and weighing about twelve tons. The microscopes were fastened to the tops of the piers by iron bolts, and all protected from temperature changes by wrapping with cotton batting and also by using them under a wooden shelter open to the north only. The metre prototype (M_{21}) was used on a car in exactly the same manner as that of using B_{17} on the base measure, and with similar adjustments.

A light much superior to the electric lights of the comparing room was obtained by whitewashing the fence and adjacent building, thus getting a very satisfactory illumination—one of the most important factors in this character of work.

Every precaution was taken to eliminate all possible sources of error, such as measuring in opposite directions with the prototype, making simultaneous observations on the ends of the bar, and repeating them, by changing positions of observers to eliminate personal equation, and being very careful to keep the bars thoroughly packed in ice so that the temperature could not change.

A careful discussion of the results show that the microscopes were satisfactorily stable, and that all the operations involved were performed with as great an accuracy as is desirable.

The precision of the results brought to light an unexpected source of systematic error, due apparently to the fact that the transverse graduation marks at the ends of B_{17} differed much in width, and to the fact that the personal equations of the observers differed widely in amount. This emphasizes the fact that the more precise base-measuring apparatuses should be standardized by the parties who are to use them in the field.

From these observations the length of B_{17} was obtained in terms

of the prototype metre with a probable error of $0{\cdot}4^{\mu}$, or $\frac{1}{12,000,000}$ part of its length.

In order to get the value of B_{17} in terms of the international metre, we must combine this probable error with that of the prototype metre as given by comparison with the international metre. The probable error of the prototype is $\pm 1{\cdot}^{\mu}$ for the length of B_{17}, whence we have $\pm 1{\cdot}1^{\mu}$ as the probable error of B_{17} in terms of the international metre, or we know its length within probably $\frac{1}{4,500,000}$ of its length.

This apparatus was used to measure a kilometre of the Holton base in southern Indiana. From four measures the length was obtained with a probable error of measurement of $\pm 0{\cdot}26$ millimetre, or $\frac{1}{3,900,000}$. Combining this with the probable error of B_{17}, the probable error of the result in terms of the international metre is $\pm 0{\cdot}37$ millimetre, or $\frac{1}{2,700,000}$. This includes all known sources of error, and also an allowance for unknown ones, or such as were not entirely compensated or corrected.

632. Duplex Base Apparatus. In 1893 the Coast and Geodetic Survey constructed an apparatus after designs of the inventor, Mr. William Eimbeck, assistant on the survey. It is a contact-slide, bimetallic apparatus, and consists of two cylindrical tubes, one of steel and the other of brass, each $\frac{3}{4}$ inch in diameter and 5 metres long. The brass tube is made of $\frac{3}{32}$-inch material, and the steel $\frac{1}{16}$-inch, in thickness. They are placed side by side, and supported at intervals of 97 centimetres in a tube about $2\frac{3}{4}$ inches in diameter. This tube is inclosed within another about 4 inches in diameter, and is supported at intervals of 1·6 metre. The inner tube, which contains the two bars, or components, as they are termed, is so arranged that it can be rotated about its axis 180°, so as to reverse the relative positions of the two components. The object of this device is to eliminate the effect of unequal heating on the sides of the apparatus while in use. The weights of the two components are made directly proportional to their specific heats and conductivities, in order that they may both receive and assume any temperature change simultaneously, as the theory of their use depends almost entirely upon such being the case. Each must

change at the same instant, and the amount of change in each must be proportional to its coefficient of expansion.

At each end of the tube a short scale is attached to the steel component and a vernier to the brass, reading tenths of millimetres. They are read through glass-covered openings in the two outside tubes.

The method of using the apparatus is to make two simultaneous measures of the base, one with the brass component and the other with the steel. Every eight or ten bar lengths the two components are reversed in position by rotating the inner tube. Readings are made on the scales at the ends of the rods whenever beginning or ending a measure, and also whenever it is necessary to set one component back in order that both may be brought into contact at the same time. From these readings we get the difference in length of the base as shown by the two metals.

The length of the two components is obtained when they are both the same length, and this is called the standard length of the apparatus. We must also know the coefficient of expansion of one of the components (say the steel) and the differential coefficient of the two metals. If now we enter the field and make a measurement in the manner described above we get the length of the base in terms of the two components.

Let D = actual length of the base;

a = the length of the two components when they are standard —i. e., equal;

N = number of bars laid;

b = sum of the set-backs during the measurement, including the difference between the first and last readings;

x = the total expansion of the steel component during the whole measure;

y = corresponding brass expansion;

E_s and E_b = the coefficients of expansion of the steel and brass components, respectively.

x and y must be directly proportional to E_s and E_b, as both the components are subjected to the same temperature changes; hence

$$\frac{x}{y} = \frac{E_s}{E_b} \text{ or } y = \frac{E_b}{E_s} x \qquad [1.]$$

also $$D = Na + x, \quad [2.]$$
$$= Na + y \pm b \quad [3.]$$
whence subtracting [3] from [2] we get
$$x - y = \pm b, \quad [4.]$$
substituting the value of y given in [1] we have
$$x - \frac{E_b}{E_s} x = \pm b$$
or $$x = \pm \frac{E_s b}{E_s - E_b}. \quad [5.]$$

That is, the correction necessary to apply to the Na, in order to get the true length of the base as shown by the steel component, is equal to the product of the set-backs into the ratio of the coefficient of expansion of the steel component divided by coefficient of expansion of the steel component minus the coefficient of expansion of the brass component.

The plus (+) sign is to be used when the steel component is set back, and the minus (—) sign when the brass component is set back; or use the plus sign when the apparatus is used at an average temperature lower than that at which it is standardized, and the minus when it is used at a higher temperature.

If care is taken to measure at such times as make the average temperature very nearly that at which the apparatus is standard, b will be very small, and any errors in the values of the coefficients of expansion of little or no consequence. As it is difficult to get a very accurate coefficient of expansion of a metal, it will be safest to always make the observations so that the expansion correction must be small, and it is for such a case that this apparatus is intended, although even where greater differences occur it is more reliable than most other forms of apparatus, for you have to depend upon only one variable quantity in place of two—i. e., the coefficient of expansion instead of the coefficient and also the determination of the actual temperature of the apparatus during the measure, and of the two the latter is considered the more unreliable.

This apparatus was used during the summer of 1896 to measure a base in Utah, giving excellent results. The inclination and alinement of these bars are obtained in the same manner as with the secondary bar described on page 63 *et seq.*

633. Accuracy Obtainable in Base Measurement. As the metre has been adopted as the international unit of length, and an International Bureau of Weights and Measures established, we must take the results of this bureau as our standard of attainable accuracy in metrological work.

From the report of the International Conference on the construction and comparison of the new metric prototype (of which the United States has two), it appears that the probable error of the result of the comparison of the prototypes was only $\pm$ 0·04 micron at the temperature at which the comparisons were made. If the coefficient of expansion be considered, however, the final estimate of the accuracy reached is stated in the following words:

"It may be concluded, therefore, that the equations of the prototypes lead to a knowledge of their absolute lengths with a mean probable uncertainty, which under the temperature conditions usual in metrological operations (i. e., between 20° and 25° C.) lies between 0·1 and 0·2$^{\mu}$ ($\frac{1}{10,000,000}$ and $\frac{1}{5,000,000}$), and at higher temperatures it may slightly exceed the last-mentioned limit."

As the above-mentioned comparisons were made under the most favorable conditions, such as uniformity of temperature, identity of material, perfect illumination (slight imperfections of the latter alone introducing very material discrepancies), we may conclude that no geodetic standard can be known with a higher degree of accuracy than one part in 5,000,000 of its length in terms of the international unit.

Since all the operations involved in the measurement of a base tend to decrease this degree of accuracy, we can safely assume that one part in 5,000,000 of its length is a higher degree of accuracy than can ever be attained in practice by the methods now employed. Some foreign base measures are published, giving very small probable errors, but it must be remembered that they do not include the error of standardizing the apparatus, being merely those incidental to the measurement itself.

The close agreement of repeated measures of a base-line shows that elimination of accidental errors has been successfully met by the various forms of apparatus in use. It is believed, too, that the

methods of observing successive lengths of the same bar, or system of bars, are sufficiently precise.

Nevertheless, constant errors exist, due mainly to a defective knowledge of the temperature of the bars, as proved by the lag of mercurial or bimetallic thermometers used on various apparatuses. These constant errors are now the principal sources of error, but it is hoped that the duplex apparatus will solve the problem, and give us an apparatus economical as well as accurate. The iced-bar apparatus will undoubtedly be our standard of accuracy, but it is too expensive for ordinary use.

Several base-lines have been measured, both in this country and abroad, using two different forms of apparatus. Although each alone may give a very small probable error, the difference between the two results is usually very much in excess of their probable errors. For example, four measures of a base with the Bessel apparatus gave a relative error of $\frac{1}{4,500,000}$, but when measured with the Brunner apparatus gave a result differing from that of the Bessel by one part in 250,000. The measurement of the Holton base in Indiana with the 100-metre steel tape, and also with the 5-metre secondary bars show the same thing. Each considered alone gives a relative error of less than $\frac{1}{1,000,000}$, yet differ by one part in 350,000. These, as well as other cases, all show large discrepancies, which seem to be due mostly to the lack of knowledge of the actual temperature of the apparatus used.

634. Degree of Accuracy necessary. This necessarily depends upon the object the base has to subserve, and on the apparatus and time available, hence no definite limit can be fixed. With the perfection of the means now on hand, a line may be readily measured with a probable error of $\frac{1}{1,000,000}$ of its length, so far as mere accuracy is concerned. On the other hand, it is not surprising if all the known and unknown errors produce finally an actual uncertainty two or three times as great as that of actual measurement. Hence, in the best order of work the apparatus must be handled with the greatest of care, and its best results obtained, especially as it is but slightly, if any, more expensive to use an apparatus as carefully as possible than with less care. The high degree of accuracy is, how-

ever, soon dissipated in the two, three, or more steps used in connecting the base with the average length of side of the triangulation system. It may be further remarked that a high degree of accuracy in the angular measures must be obtained if an error of $\frac{1}{150,000}$ is not to be exceeded throughout the triangulation. In tertiary triangulation an average uncertainty, or lowest limit, of $\frac{1}{10,000}$ may be satisfactory for the special purpose, and for secondary or intermediate triangulation $\frac{1}{100,000}$ to $\frac{1}{50,000}$ may be suitable, and the degree of accuracy for the base supporting such work should be graduated accordingly, always being certain to get a probable error very much less than that allowable in the triangulation, so as not to increase the errors of the triangulation by that means.

In connecting the base-line with the main scheme great care must be given to the minute centering of the instrument and signals at the various stations.

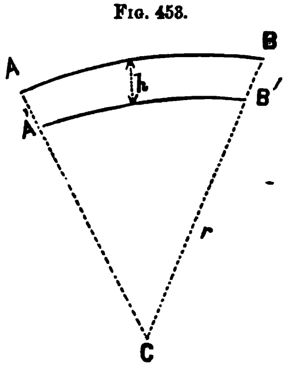
635. Reducing the Base to the Level of the Sea. Let $AB = a$ be the measured base, and $A'B' = x$ the base reduced to the level of the sea, h the height of the measured base above the level of the sea, and ρ the radius of the earth to the level of the sea. Then we have

$$\rho + h : \rho :: a : x.$$

$$\therefore x = a\frac{\rho}{\rho + h}.$$

$$a - x = \frac{a\,h}{\rho + h} = \frac{\dfrac{a\,h}{\rho}}{1 + \dfrac{h}{\rho}} = \frac{a\,h}{\rho}\left(1 + \frac{h}{\rho}\right)^{-1}$$

Fig. 458.



Developing by the binomial formula, we get

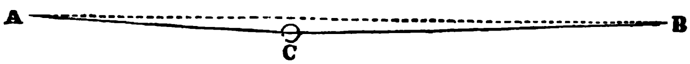
$$a - x = a\frac{h}{\rho} - a\frac{h^2}{\rho^2} + a\frac{h^3}{\rho^3} -, \text{etc.}$$

As h is very small in comparison with ρ, the first term of the correction is generally sufficient.

636. A Broken Base. When the angle C is very obtuse, the lines A C and C B being measured, and forming nearly a straight line, the length of the line A B is found thus: Naming the lines, as

is usual in trigonometry, by small letters corresponding to the capital letters at the angles to which they are opposite, and letting

FIG. 454.



K = the number of minutes in the supplement of the angle C, we shall have

$$A\,B = c = a + b - 0{\cdot}000000042308 \times \frac{a\,b\,K^2}{a+b}.$$

$$\text{Log. } 0{\cdot}000000042308 = 2{\cdot}6264222 - 10.$$

Proof. Art. 12, Theorem III [Trigonometry, Appendix A], gives, $\cos. C = \frac{a^2+b^2-c^2}{2\,a\,b}$; or $c^2 = a^2 + b^2 - 2\,a\,b.\cos. C$. This becomes [Trig., Art. 6], K being the supplement of C, $c^2 = a^2 + b^2 + 2\,a\,b.\cos. K$. The series [Trig., Art. 5] for the length of a cosine gives, taking only its first two terms, since K is very small, $\cos. K = 1 - \frac{1}{2}K^2$. Hence,

$$c^2 = a^2 + b^2 + 2\,a\,b - a\,b\,K^2 = (a+b)^2 - a\,b\,K^2 = (a+b)^2\left(1 - \frac{a\,b\,K^2}{(a+b)^2}\right);$$

whence $$c = (a+b)\sqrt{\left(1 - \frac{a\,b\,K^2}{(a+b)^2}\right)}.$$

Developing the quantity under the radical sign by the binomial theorem, and neglecting the terms after the second, it becomes

$$1 - \tfrac{1}{2}\cdot\frac{a\,b\,K^2}{(a+b)^2} +, \text{ etc.}$$

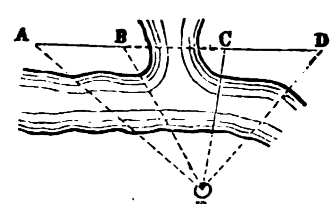
Substituting for K seconds, K . sin. 1″ [Trig., Art. 5], and performing the multiplication by $a + b$, we obtain

$$c = a + b - \frac{a\,b\,K^2.(\sin.1'')^2}{2\,(a+b)}. \quad \text{Now, } \frac{(\sin.1'')^2}{2} = 0{\cdot}0000000423079;$$

whence the formula, $c = a + b - 0{\cdot}000000042308 \times \frac{a\,b\,K^2}{a+b}$.

637. Problem to interpolate a Base. *Four inaccessible objects,* A, B, C, D, *being in a right line, and visible from only one point,* E, *it is required to determine the distance between the middle points,* B *and* C, *the exterior distances,* A B *and* C D, *being known.*

FIG. 455.



Let $A\,B = a$, $C\,D = b$, $B\,C = x$; $A\,E\,B = P$, $A\,E\,C = Q$, $A\,E\,D = R$.

Calculate an auxiliary angle, K, such that

$$\text{tang.}^2\, K = \frac{4\,a\,b}{(a-b)^2} \cdot \frac{\sin.\ Q \,.\, \sin.\ (R-P)}{\sin.\ P \,.\, \sin.\ (R-Q)}.$$

Then is $x = -\frac{a+b}{2} \pm \frac{a-b}{2\,.\,\cos.\ K}$.

Of the two values of x, the positive one is alone to be taken.

This problem is used when a portion of a base-line passes over water, etc.

Proof. In Fig. 455, produce A D to some point F. The exterior angles, $EBC = A + P$; $ECD = A + Q$; $EDF = A + R$. The triangle ABE gives $\frac{BE}{a} = \frac{\sin.\ A}{\sin.\ P}$. The triangle ACE gives $\frac{CE}{a+x} = \frac{\sin.\ A}{\sin.\ Q}$.

Dividing member by member, we get $\frac{BE}{CE} = \frac{a\,.\,\sin.\ Q}{(a+x)\sin.\ P}$.

In the same way the triangle BED and CED give $\frac{BE}{b+x} = \frac{\sin.\ (A+R)}{\sin.\ (R-P)}$; and $\frac{CE}{b} = \frac{\sin.\ (A+R)}{\sin.\ (R-Q)}$. Whence, as before, $\frac{BE}{CE} = \frac{(b+x)\sin.\ (R-Q)}{b\,.\,\sin.\ (R-P)}$.

Equating these two values of the same ratio, we get

$$\frac{a\,.\,\sin.\ Q}{(a+x)\sin.\ P} = \frac{(b+x)\sin.\ (R-Q)}{b\,.\,\sin.\ (R-P)}; \text{ and thence}$$

$$\frac{a\,b\,.\,\sin.\ Q\,.\,\sin.\ (R-P)}{\sin.\ P\,.\,\sin.\ (R-Q)} = (a+x)(b+x) = ab + (a+b)x + x^2.$$

To solve this equation of the second degree, with reference to x, make

$$\tan.^2 K = \frac{4\,a\,b}{(a-b)^2} \cdot \frac{\sin.\ Q\,(\sin.\ R-P)}{\sin.\ P\,(\sin.\ R-Q)}.$$

Then the first member of the preceding equation $= \frac{1}{4}\,.\,(a-b)^2 \tan.^2 K$, and we get $\quad x^2 + (a+b)x = \frac{1}{4}(a-b)^2\,.\,\tan.^2 K - ab$,

and

$$\begin{aligned} x &= -\tfrac{1}{2}(a+b) \pm \sqrt{[\tfrac{1}{4}(a-b)^2\,.\,\tan.^2 K - a\,b + \tfrac{1}{4}(a+b)^2]}, \\ &= -\tfrac{1}{2}(a+b) \pm \sqrt{[\tfrac{1}{4}(a-b)^2\,.\,\tan.^2 K + \tfrac{1}{4}(a-b)^2]}, \\ &= -\tfrac{1}{2}(a-b) \pm \tfrac{1}{2}(a-b)\sqrt{(\tan.^2 K + 1)}. \end{aligned}$$

Or, since $\sqrt{(\tan.^2 K + 1)} = \text{secant } K = \frac{1}{\cos.\ K}$, we have $x = -\frac{a+b}{2} \pm \frac{a-b}{2\,.\,\cos.\ K}$.

When $a = b$, or when the two known parts are equal to each other, the above solution is indeterminate. For this case put

$$\tan.^2 K' = \frac{a\,b\sin.\ Q\sin.\ (R-P)}{\sin.\ P\,.\,\sin.\ (R-Q)},$$

and the solution gives

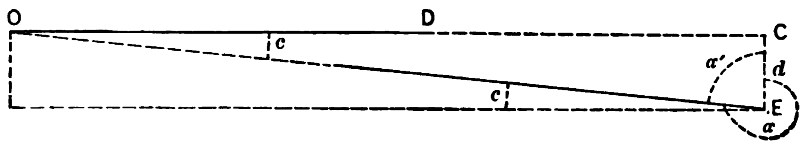
$$x = -\tfrac{1}{2}(a+b) \pm \sqrt{\tan.^2 K' + \frac{(a-b)^2}{4}}.$$

If $a = b$, this becomes

$$x = -\tfrac{1}{2}(a+b) \pm \tan.\ K'.$$

638. Eccentric Reduction. Whenever the instrument or the signal has to be mounted away from the station for any reason, the observations must be reduced to what they would have been if the instrument, or signal, had been placed directly over the station mark. In order to make this correction the eccentric distance and direction with reference to the observed lines are measured and recorded with the observations.

Fig. 456.



Let C = center mark of station;

E = eccentric station;

d = eccentricity;

D = distance between C and object observed, O;

α = angle between C and object observed;

c = angular correction to reduce to center.

Then, in the triangle O C E (Fig. 456) we have

$$\frac{\sin. c}{d} = \frac{\sin. \alpha}{D},$$

or, c (in $''$) $= \dfrac{d \sin. \alpha}{D \sin. 1''}$ (since c is a small angle).

A preliminary computation of the triangles will usually give the length of the sides with sufficient accuracy for this computation, unless the eccentricity is quite large or extreme accuracy is desired. In such cases it will be necessary to make a new computation, using the corrected angles for the determination of the triangle sides.

The following example shows a convenient form for making this computation:

Example. Station, insane asylum, eccentric station.

d = 14 feet 4·87 inches = 172·87 inches,	log.	= 2·23772
reduction inches to metres,	"	= 8·40483
sin. 1″ a. c.	"	= 5·31442

$$\log. \frac{d}{\sin. 1''} = 5{\cdot}95697$$

STATION.	α	LOG. SIN. α	LOG. D.	LOG. $\frac{\text{SIN. } \alpha}{\text{D.}}$	LOG. $\frac{d \text{ SIN. } \alpha}{\text{D SIN. } 1''}$	CORR'N.
Insane asylum center.	00° 00′					
Morgan	119° 18′	9·9406	4·1083	5·8323	1·7893	+ 61·6″
Minoma	172° 50′	9·0961	4·0252	5·0709	1·0279	+ 10·7″
Std. pipe	212° 38′	9·7318$_n$	3·9825$_n$	5·7493$_n$	1·7063$_n$	− 50·9″
Sec. Prs. Ch	237° 07′	9·9242$_n$	3·8598$_n$	6·0644$_n$	2·0214$_n$	−105·1″

To obtain the correction to any angle, if angles were observed, we have merely to subtract the correction opposite the first station from that opposite the other and apply the resulting difference as the correction to this angle, as shown in the following arrangement:

OBJECTS OBSERVED.	OBSERVED ANGLES	CORRECTION.	REDUCED ANGLES.
Insane asyl. center, std. pipe	212° 37′ 30·0″		
Morgan, std. pipe..	93° 19′ 8·6″	(− 61·6″−50·9″)=−1′ 52·5″	93° 17′ 16·1″
Minoma, std. pipe..	39° 47′ 45·2″	(− 10·7″−50·9″)=−1′ 1·6″	39° 46′ 43·6″
Sec. Prs. Ch. Minoma	295° 42′ 51·7″	(+105·1″+10·7″)=+1′ 55·8″	295° 44′ 47·5″

639. Reduction of Horizontal Directions to Sea Level. Whenever great precision is desired in primary work a small correction should be applied to the angles for elevation of station above sea level. It is due to the fact that the vertical of any point when viewed from another point does not appear vertical; hence, if B′ is the projection upon the spheroidal surface of a signal B at a height h above B′, then to an observer at A, B′ and B are not in the same vertical plane unless A and B happen to be in the same latitude. The angular correction to be made is given by, *

$$c = \frac{e^2 h}{2\rho} \text{ sin. } 2 \text{ A cos.}^2 \phi,$$

when e = eccentricity of the earth $= \left(\frac{a^2 - b^2}{a^2}\right)^{\frac{1}{2}}$ (a being equatorial semi-axis and b polar semi-axis), ρ = mean radius of curvature of the earth, A the azimuth of the line of sight considered, counting from S. to W. to N. to E., and ϕ the latitude of the line.

The correction is positive for a line whose azimuth is in the first

* "Geodesy," Clarke, p. 113. "Vermessungskunde," Jordan, third edition, vol. iii, p. 363.

or third quadrants, and negative when the azimuth is in the second or fourth quadrants. It is a maximum on lines making angles of 45° with the meridians or parallels, and is zero when the line coincides with a meridian or parallel. In the United States this correction can not exceed a tenth of a second for each kilometre of elevation, and even on the equator, where it is greatest, can not exceed 0·15″ per kilometre of height. It may therefore be neglected in all work excepting primary triangulation, and is often neglected in this when the elevations above sea level are not great, as it is more or less compensating in a chain of triangulation.

640. Adjustment. After correcting the observed angles or directions for eccentricity of signal or instrument, phase whenever appreciable, and reducing them to what they would have been if made at sea level, it will be found that small discrepancies will result wherever there are any conditions to be filled, as it is practically impossible to get the true value of any continuous quantity, such as an angle or line.

When good work has been done these discrepancies should be small, and their size affords a criterion of the accuracy of the work. In order to free the results from these discrepancies and obtain the most probable values, the observed quantities are adjusted by the method of least squares—i. e., small corrections are applied to them so that no discrepancies will be shown in any of the conditions, and also the most probable values obtained according to the theory of probabilities. Experience has shown that the least square adjustment is the best that can be used, and it therefore is applied to nearly all observations in order to make them accordant.

In any triangulation net with a single measured base, in which the sides are to be computed from this base through the intervening triangles, all contradictions among the measured angles may be removed and a consistent figure obtained if the angles are adjusted so as to satisfy two classes of conditions:

1. Those arising at each station from the relation of the angles to one another at that station. These are known as *local* or *station* conditions.

2. Those arising from the geometrical relations necessary to form closed figures.

(*a*) That the sum of the angles of each triangle must equal two right angles plus the spherical excess of the triangle.

(*b*) That the length of any side, as computed from the base, should be the same whatever route is chosen.

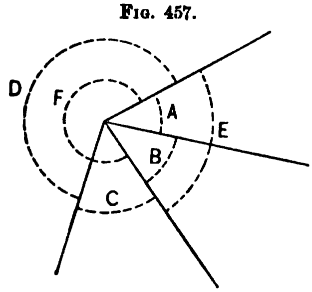
These are called *general* or *figure* conditions.

The number of conditions to be satisfied will depend upon the measurements made. Each condition can be stated in the form of an equation in which the most probable value of the measured quantities are the unknowns. As the number of equations is usually less than the number of unknowns, an infinite number of solutions is possible. The problem, then, is to select the most probable values from this infinite number of solutions.

Whenever great accuracy is desired all the equations of condition possible, both local and general, throughout the whole net of triangulation, should be formed and solved simultaneously. Ordinarily, however, this method is too laborious, and not warranted by the limited number of observations obtained. Hence the local or station adjustments are made first, and afterward the general or figure adjustments applied in such a way as not to disturb the previous adjustment, although the individual values of the angles may have been changed.

641. Local or Station Adjustments. The form of reduction for local adjustment depends upon the method employed in making the observations, whether each angle was observed independently of the others, as when a repeating theodolite is used, or whether directions were observed with a non-repeating theodolite.

FIG. 457.



The first is known as the method of *independent angles*, and the second as the method of *directions*.

642. Independent Angles. At any one station only two kinds of conditions are possible:

(*a*) That an angle can be formed from two or more others; and

(*b*) That the sum of the angles around the horizon should equal 360°—e. g., in Fig. 457 the sum of the measured values of the angles A and B should be equal to the measured value of E, and we should have the equation,

$$A + B = E;$$

also
$$C + D = F,$$

and
$$A + B + C + D = 360°.$$

In general, if M_1, M_2, etc.... M_n denote the single measured angles, and v_1, v_2, ... v_n their most probable corrections, then if any of the angles as M_h, M_k, can be formed from others, we have by equating the measured and computed values the local condition equations,

$$\begin{aligned} M_h + v_h &= M_1 + v_1 + M_2 + v_2 + \dots \\ M_k + v_k &= M_1 + v_1 + M_2 + v_2 + \dots \end{aligned}$$

or,

$$\begin{aligned} v_1 + v_2 + \dots - v_h &= l_h \\ v_1 + v_2 + \dots - v_k &= l_k, \end{aligned}$$

l_h and l_k .... being the discrepancies developed in the various condition equations.

The solution is generally carried out by the method of correlates, as in the following example. The theory of the solution may be found in Wright's "Treatise on the Adjustment of Observations," "United States Coast and Geodetic Survey Report" for 1854, etc.

643. Number of Local Equations at a Station. If s stations are observed from a station, the number of angles necessary to be measured to determine all the angles that can be formed at the station occupied is $s - 1$. Hence, if an additional angle were measured its value could be determined in two ways: from the direct measurement and from the $s - 1$ measures. These two conflicting values give rise to a condition equation. If, therefore, there are n angles observed at a station, and s is the number of stations observed upon, the number of condition equations will be shown by

$$n - (s - 1) = n - s + 1.$$

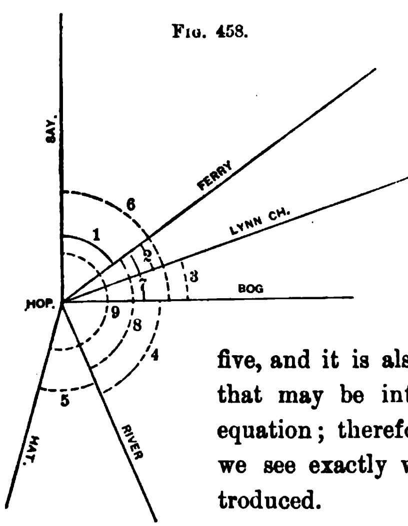
Example.—Station Hop (Fig. 458) has been reoccupied and all the following angles measured, excepting Say-Hat, which was meas-

ured previously and adjusted then. As it is desirable in this case to not disturb the previous adjustment, the angle Say-Hat is considered correct, and hence does not enter the computation.

ANGLES.	OBSERVED.	p.	CORRECTION.	ADJUSTED.
Say to Ferry	53° 12′ 01·5″	5	$v_1 = +1{\cdot}1''$	53° 12′ 02·6″
Ferry to Lynn Ch	17° 27′ 36·7″	1	$v_2 = -1{\cdot}8''$	17° 27′ 34·9″
Lynn Ch. to Bog	18° 03′ 01·7″	1	$v_3 = -1{\cdot}9''$	18° 02′ 59·8″
Bog to River	68° 01′ 31·3″	2	$v_4 = -8{\cdot}7''$	68° 01′ 22·6″
River to Hat	39° 14′ 01·7″	4	$v_5 = -1{\cdot}0''$	39° 14′ 00·7″
Say to Hat	195° 58′ 00·6″			195° 58′ 00·6″
Say to Bog	88° 42′ 39·8″	4	$v_6 = -2{\cdot}5''$	88° 42′ 37·3″
Ferry to Bog	35° 30′ 36·2″	4	$v_7 = -1{\cdot}5''$	35° 30′ 34·7″
Ferry to River	103° 31′ 54·0″	4	$v_8 = +3{\cdot}3''$	103° 31′ 57·3″

In order to find how many conditions there are we apply the formula given above,

FIG. 458.



$s = 6$, $n = 9$; hence $n - s + 1 = 4$, the number of condition equations that we must solve in order to remove all the discrepancies from the above angles. If the angles are arranged as shown above, we see that there can be no conditions among the first five, and it is also evident that any other angles that may be introduced will give a condition equation; therefore by the above arrangement we see exactly what conditions have been introduced.

Condition Equations.

$$\begin{array}{lr}
 & +\ 53^\circ\ 12'\ 01{\cdot}5'' + v_1 \\
 & +103^\circ\ 31'\ 54{\cdot}0'' + v_8 \\
 & +\ 39^\circ\ 14'\ 01{\cdot}7'' + v_5 \\
\hline
 & 195^\circ\ 57'\ 57{\cdot}2'' \\
\text{Say-Hat} & 195^\circ\ 58'\ 00{\cdot}6'' \\
\hline
 & -\ 3{\cdot}4
\end{array}
\qquad
\begin{array}{r}
68^\circ\ 01'\ 31{\cdot}3'' + v_4 \\
35^\circ\ 30'\ 36{\cdot}2'' + v_7 \\
\hline
103^\circ\ 32'\ 07{\cdot}5'' \\
103^\circ\ 31'\ 54{\cdot}0'' - v_8 \\
\hline
+\ 13{\cdot}5 \\
\ldots\ \text{etc.}
\end{array}$$

Hence

$$
\begin{aligned}
&1.\quad v_1 + v_5 + v_8 - 03{\cdot}4 = 0\\
&2.\quad v_4 + v_7 - v_8 + 13{\cdot}5 = 0\\
&3.\quad v_1 - v_6 + v_7 - 02{\cdot}1 = 0\\
&4.\quad v_2 + v_3 - v_7 + 02{\cdot}2 = 0
\end{aligned}
$$

CORRELATE EQUATIONS.

v	$^4/_p$	c_1.	c_2.	c_3.	c_4.	VALUES.	VALUES $\times\, ^4/_p$.
1	0·8	+1		+1		+1·41	+1·1
2	4·0				+1	−0·46	−1·8
3	4·0				+1	−0·46	−1·9
4	2·0		+1			−4·38	−8·7
5	1·0	+1				−1·05	−1·0
6	1·0			−1		−2·46	−2·5
7	1·0		+1	+1	−1	−1·46	−1·5
8	1·0	+1	−1			+3·33	+3·3

NORMAL EQUATIONS.

N	c_1.	c_2.	c_3.	c_4.	
− 3·4	+2·8	−1·0	+0·8		$c_1 = -1{\cdot}05$
+13·5		+4·0	+1·0	−1·0	$c_2 = -4{\cdot}38$
− 2·1			+2·8	−1·0	$c_3 = +2{\cdot}46$
+ 2·2				+9·0	$c_4 = -0{\cdot}46$

Solving these equations by the direct method, using Crelle's multiplication tables, we have

N.	c_1.	c_2.	c_3.	c_4.	CHECK Σ.	FACTORS.
− 3·4	+2·8	−1·0	+0·8		− 0·8	(1) (for 2)+1·00+2·8 = +0·387
						(" 3)−0·8 +2·8 = −0·286
+13·5		+4·0	+1·0	−1·0	+16 5	(" 4) 0·0 +2·8 = 0·000
− 1·21		−0·36	+0·29		− 0·29	
+12·29		+3·64	+1·29	−1·0	+16·21	(2) (for 3)−1·29+3·64 = −0·355
						(" 4)+1·00+3·64 = +0·275
− 2·10			+2·80	−1·00	+ 1·50	
+ 0·97			−0·23		+ 0·23	
− 4·35			0·46	+0·36	− 5·75	
− 5·49			+2·11	−0·64	− 4·02	(3) (for 4)+0·64+2·11 = +0·303
+ 2·20				+9·00	+ 9·20	
+ 3·38				−0·28	+ 4·46	
− 1·66				−0·19	− 1·22	
+ 3·92				+8·53	+12·44	(4)

SUBSTITUTION FOR FINAL VALUES OF C_1 C_2.

$+3{\cdot}92$				$+8{\cdot}53C_4$	$-\frac{3\cdot92}{8\cdot53}=-0{\cdot}460=C_4$
$-5{\cdot}49$			$+2\ 11\,C_3$	$+0{\cdot}29$	$+\frac{5\cdot20}{2\cdot11}=+2{\cdot}46=C_3$
$+12{\cdot}29$		$+3{\cdot}64\,C_2$	$+3{\cdot}18$	$+0{\cdot}46$	$-\frac{15\cdot93}{3\cdot64}=-4{\cdot}377=C_2$
$-3{\cdot}4$	$+2{\cdot}80\,C_1$	$+4{\cdot}38$	$+1{\cdot}97$		$-\frac{2\cdot95}{2\cdot80}=-1{\cdot}05=C_1$

For convenience in making the figure adjustment, we may arrange these angles so as to give the directions of the various stations referred to one of them, as Say,

Say,	00° 00′ 00·0″
Ferry,	53° 12′ 02·6″
Lynn Ch.,	70° 39′ 37·5″
Bog,	88° 42′ 37·3″
River,	156° 43′ 59·9″
Hat,	195° 58′ 00·6″

644. Local Adjustment for Directions. When directions are observed we have a number of series on various signals all referred to the initial station. If all the stations are observed in each series, and equally well measured, we get the values for the various directions by simply taking the mean of all the series for each station. But whenever a station is lost in any series we can no longer take the mean directly, but must introduce an adjustment.

In the first series, let X denote the most probable value of the angle between the zero of the limb of the instrument and the direction of the initial station; then, if M_1', M_1'', ... denote the readings of the limb for the different signals observed in this series, and v_1', v_1'', ... the most probable corrections to these readings (each so-called reading is, of course, the mean of two, one direct and one reversed, as explained on page 54), we have the observation equations,

$$\begin{aligned}
X_1 - \phantom{A - {}} M_1' &= v_1' \\
X_1 + A - M_1'' &= v_1'' \\
X_1 + B - M_1''' &= v_1''' \\
\cdot \quad \cdot \quad &\cdot \quad \cdot
\end{aligned}$$

A, B, ... being the most probable values for the direction of each signal referred to the initial station.

The zero of the limb being changed in the next series, we have

$$\begin{aligned} X_2 \qquad - M_2' &= v_2' \\ X_2 + A - M_2'' &= v_2'' \\ X_2 + B - M_2''' &= v_2''' \\ \cdot \quad \cdot \quad \cdot \quad & \cdot \end{aligned}$$

and so on for each position.

If $p_1', p_1'', \ldots . p_2', p_2'', \ldots .$ denote the weights of the measured directions of the several series, we may write the normal equations at once, viz.:

$$\begin{array}{llll} [p_1]\,X_1 & & +p_1''\,A + p_1'''\,B + \ldots . & = [p_1\,M_1] \\ & [p_2]\,X_2 & +p_2''\,A + p_2'''\,B + \ldots . & = [p_2\,M_2] \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ p_1''\,X_1 + p_2''\,X_2 + \ldots . + [p'']\,A & & & = [p''\,M''] \\ p_1'''\,X_1 + p'''\,X_2 + \ldots . \quad + [p''']\,B & & & = [p'''\,M'''] \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \end{array}$$

from which the unknowns may be found.

To shorten the work, however, we may assume approximate values for each direction, and then take the difference between this assumed value and each observed value for the direction. These differences are represented by $m_1'', m_1''', \ldots . m_2'', m_2''', \ldots .$ etc.

$$\begin{array}{lll} \text{Where} & m_1'' = M_1'' - M_1' - A', & m_2'' = M_2'' - M_2' - A' \\ & m_1''' = M_1''' - M_1' - B', & m_2''' = M_2''' - M_1' - B' \\ & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

A′, B′, ... being the assumed approximate values for the directions. The normal equations then become, (A), (B) ...., being the most probable corrections to A′, B′, ...

$$\begin{aligned} &\left\{[p''] - \frac{p_1''}{[p_1]}\,p_2'' - \frac{p_2''}{[p_2]}\,p_2'' - \ldots .\right\}(A) \\ &\qquad + \left\{-\frac{p_1''}{[p_1]}\,p_1''' - \frac{p_2''}{[p_2]}\,p_2''' \ldots .\right\}(B) + \ldots . \\ &\qquad = [p''\,m''] - \frac{p_1''}{[p_1]}\,[p_1\,m_1] - \frac{p_2''}{[p_2]}\,[p_2\,m_2] \ldots . \\ &\left\{-\frac{p_1''}{[p_1]}\,p_1''' - \frac{p_2''}{[p_2]}\,p_2''' - \ldots .\right\}(A) + \\ &\qquad \left\{[p'''] - \frac{p_1'''}{[p_1]}\,p_1''' - \frac{p_2'''}{[p_2]}\,p_2''' - \ldots .\right\}(B) + \ldots . \\ &\qquad = [p'''\,m'''] - \frac{p_1'''}{[p_1]}\,[p_1\,m_1] - \frac{p_2'''}{[p_2]}\,[p_2\,m_2] \ldots . \\ &\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

These equations may be written,

$$\underline{[a\,a]}\,(A) + \underline{[a\,b]}\,(B) + \ldots. = \underline{[a\,l]}$$
$$\underline{[a\,b]}\,(A) + \underline{[b\,b]}\,(B) + \ldots. = \underline{[b\,l]}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

when $\underline{[a\,a]}$ $\underline{[a\,b]}$ .... are merely symbols.

If we arrange the observations in groups, each containing all the series in which the same signals were observed, we may still further shorten the work, as all the observations in a group may be considered of equal weight; hence,

$$p_1' = p_1'' = \ldots. = {}_1p$$
$$p_2' = p_2'' = \ldots. = {}_1p$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

and $\qquad [p_1] = n_s'\,{}_1p \qquad [p_2] = n_s'\,{}_1p$

for first group, and similarly for the others, n_s' being the number of stations observed in this group.

If n', n'', ... denote the number of series in the several groups, we may write the coefficients of the normal equations as follows:

$$\underline{[a\,a]} = [p''] - \frac{n'}{n_s'}\,{}_1p'' - \frac{n''}{n_s''}\,{}_2p'' - \ldots.$$
$$\underline{[a\,b]} = -\frac{n'}{n_s'}\,{}_1p''\,{}_1p''' - \frac{n''}{n_s''}\,{}_2p''\,{}_2p''' - \ldots.$$
$$\underline{[a\,l]} = [p''\,m''] - \frac{[m_1]}{n_s'}\,{}_1p'' - \frac{[m_2]}{n_s''}\,{}_2p'' - \ldots.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

These weights for the signals observed may be taken as unity, and for the stations not observed as zero. Having found m', m'', ... by taking the differences between the observed quantities and the assumed approximate values, it is convenient to arrange the formation of the normal equations according to the following scheme:

NO. OF GROUP.	$\frac{n}{n_s}$	p	$p''\,m''$	$p'''\,m'''$	....	sum.	$\frac{\text{sum}}{n_s}$
1 2 3							
		$[p]$	$[p''\,m'']$	$[p'''\,m''']$	....	$[p\,m]$	

from which we may write the coefficients of the normal equations at sight.

Checks of Normal Equations.—1. The sum of all the coefficients of (A), (B), ... in the normal equations should be equal to half the number of observations, less half the number of series.

2. The sum of the

$$\underline{[a\,l]} + \underline{[b\,l]} + \underline{[c\,l]} + \ldots\ldots = \underline{[w\,l]}$$

where $\underline{[w\,l]}$ is formed same as $\underline{[a\,l]}$, $\underline{[b\,l]}$, ...

Example. The following are the values for $M_1'' - M_1'$, $M_1''' - M_1'$, ... as obtained from the abstract of the observations at this station:

STATION CLARK.

SPEAR.	HUM.	FORK.
° ′ ″	° ′ ″	° ′ ″
00 00 00·00	24 09 35·70	78 26 08·55
·00	39·40	09·60
·00	36·23	09·33
·00	33·55	10·45
00 00 00·00		78 26 10·20
·00		11·08
00 00 00·00	24 09 36·10	
00·00	37·40	
·00	36·53	
·00	38·15	
·00	39·00	
	00 00 00·00	54 16 31·85
	00·00	31·94
	·00	32·03
	·00	36·19

Assuming the approximate values,

Spear, 00° 00′ 00·00″
Hum, 24° 09′ 36·90″ + (A)
Fork, 78° 26′ 09·90″ + (B)

we get the following values for the m's:

$p'\,m'$	$p''\,m''$	$p'''\,m'''$	SUMS.
$n'=4$ 0·00"	−1·20"	−1·35"	
$n_s'=3$ 0·00	$_1p''=1$ +2·50	$_1p'''=1$ −0·30	
0·00	−0·67	−0·57	
0·00	−3·35	+0·55	
0·00	−2·72	−1·67	−4·39"
$n'=2$ 0·00		$_2p'''=1$ +0·30	
$n_s'=2$ 0·00	$_2p''=0$	+1·13	
0·00		+1·43	+1·43
00·00	−0·80		
$n'=5$ 0·00	+0·50		
$n_s'=2$ 0·00	$_3p''=1$ −0·37	$_3p'''=0$	
0·00	+1·25		
0·00	+2·10		
0·00	+2·68		+2·68
$n'=4$	0·00	−1·15	
$n_s'=2$	$_4p''=1$ 0·00	$_4p'''=1$ −1·06	
	0·00	−0·97	
	0·00	+3·13	
	0·00	−0·05	−0·05
$[p'\,m']=00·00$ $[p']=11$	$[p''\,m'']=-0·04$ $[p'']=13$	$[p'''\,m''']=-0·29$ $[p''']=10$	$[p\,m]=-0·33$

Then we may form the table:

NO. OF GROUP.	$\frac{n}{n_s}$	n	$p'\,m'$	$p''\,m''$	$p'''\,m'''$	SUM.	$\frac{\text{SUM}}{n_s}$
1	$4/_3$	4	0	−2·72	−1·67	−4·39	−1·463
2	$2/_2$	2	0		+1·43	+1·43	+0·715
3	$5/_2$	5	0	+2·68		+2·68	+1·340
4	$4/_2$	4	0	0·00	−0·05	−0·05	−0·025
Sums.	..	15	..	−0·04	−0·29	−0·33	Check.

Hence the coefficients of the normal equations are:

$$[a\,a] = 13 - \tfrac{4}{3} - \tfrac{5}{2} - \tfrac{4}{2} = +7\tfrac{1}{6};$$

$$[a\,b] = \quad - \tfrac{4}{3} - \tfrac{4}{2} \quad = -3\tfrac{1}{3};$$

$$[b\,b] = 10 - \tfrac{4}{3} - \tfrac{2}{2} - \tfrac{4}{2} = +5\tfrac{2}{3};$$

$$[a\,l] = -0·04 + 1·463 - 1·34 + 0·025 = +0·108$$

$$[b\,l] = -0·29 + 1·463 - 0·715 + 0·025 = +0·483$$

$$[wl] = -1·463 + 0·715 + 1·34 \qquad = +0·592$$

$$\text{Check 1,} = [aa] + [ab] + [bb] = 9\tfrac{1}{2}$$

$$\frac{\text{Number of observations} - n}{2} = \frac{34 - 15}{2} = 9\tfrac{1}{2}$$

$$\text{Check 2,} = [al] + [bl] = 0{\cdot}591 = [wl].$$

Substituting the above coefficients in the normal equations, we have

$$7\tfrac{1}{6}(\text{A}) - 3\tfrac{1}{3}(\text{B}) = +0{\cdot}108$$

$$-3\tfrac{1}{3}(\text{A}) + 5\tfrac{2}{3}(\text{B}) = +0{\cdot}483$$

Hence

$$(\text{A}) = +0{\cdot}075$$

$$(\text{B}) = +0{\cdot}130$$

and the resulting adjusted local directions are

Spear, 00° 00′ 00·000″
Hum, 24° 09′ 36·975″
Fork, 78° 26′ 10·030

There are several approximate methods of adjustment both for angles and directions, but we have not space to reproduce them here. If desired, several may be found in Wright's "Treatise on the Adjustment of Observations."

645. General or Figure Adjustment. Having satisfied all the local or station conditions of the triangulation, we next eliminate the discrepancies of the figures—i. e., those arising from the geometrical relations necessary to form closed figures. These are of two forms:

1. That the sum of the angles of each triangle should be equal to 180°, increased by the spherical excess.

2. That the length of any side, as computed from the base, should be the same whatever route is chosen.

The equations arising from these conditions are known as *angle* and *side* equations respectively.

646. Spherical or Spheroidal Excess of Triangles. We know that the sum of the angles of any triangle on a sphere, or spheroid, is greater than 180°. On a sphere this excess bears the same relation to eight right angles as the area of the triangle bears to the area of the whole sphere. Let r = radius of the sphere, ϵ = the excess of the triangle, then we have

$$\frac{\epsilon}{4\pi} = \frac{\text{area}}{4r^2\pi}$$

hence $\epsilon = \frac{\text{area}}{r^2 \sin. 1''}$ where we divide by sin. 1″ to get ϵ in seconds of arc. All geodetic triangles are small compared with the whole globe, and we may therefore express the area with sufficient accuracy by $\frac{1}{2}a_1\, b_1 \sin. C_1$ where a_1 and b_1 are two sides, and C_1 the included angle, of the triangle.

We then have

$$\epsilon = \frac{a_1\, b_1 \sin. C_1}{2\, r^2 \sin. 1''}$$

This is the expression for the excess of a triangle on a true sphere, however, and we must change it to fit the terrestrial spheroid. We do this with sufficient accuracy by referring to an osculating sphere the radius of which is $\sqrt{\text{R N}}$, where R is the radius of curvature in the meridian and N the radius of curvature in the prime vertical, at the center of the triangle. These are respectively,

$$\text{R} = \frac{a\,(1 - e^2)}{(1 - e^2 \sin.^2 \phi)^{\frac{3}{2}}}, \qquad \text{N} = \frac{a}{(1 - e^2 \sin.^2 \phi)^{\frac{1}{2}}}$$

using the notation of Art. 654.

Hence for the spheroidal triangle upon the earth we have

$$\epsilon = \frac{a_1\, b_1 \sin. C_1}{2\, \text{R N} \sin. 1''} = \frac{a_1\, b_1 \sin. C_1}{2\, a^2\,(1 - e^2) \sin. 1''}\,(1 - e^2 \sin.^2 \phi)^2,$$

or $\epsilon = a_1\, b_1 \sin. C_1\, m$ *

where $m = \frac{(1 - e^2 \sin.^2 \phi)^2}{2\, a^2\,(1 - e^2) \sin. 1''}.$

The logarithm of m for various latitudes is given in the table on the following page.

m must be taken for the middle latitude of the triangle or for the mean of the latitudes of the three stations.

Before we can compute the excess we must make a preliminary computation to find $a_1\, b_1$ and the latitudes. The values found by using the unadjusted angles will be close enough for this purpose, and the latitudes need only be computed to the nearest minute.

* This is sufficiently accurate for all ordinary triangles, where Legendre's theorem is applicable. For very large triangles we must use other formulas for spherical excess. See, for example, Helmert, "Theorieen d. höheren Geodesie," vol. i, p. 362.

TABLE OF LOG. *m*.

LATITUDE.	LOG. *m*.	LATITUDE.	LOG. *m*.	LATITUDE.	LOG. *m*.	LATITUDE.	LOG. *m*.
° ′		° ′		° ′		° ′	
18 00	1·40639	33 00	1·40520	48 00	1·40369	63 00	1·40227
18 30	636	33 30	516	48 30	364	63 30	223
19 00	632	34 00	511	49 00	359	64 00	219
19 30	629	34 30	506	49 30	354	64 30	215
20 00	626	35 00	501	50 00	349	65 00	210
20 30	623	35 30	496	50 30	344	65 30	207
21 00	619	36 00	491	51 00	339	66 00	203
21 30	616	36 30	486	51 30	334	66 30	199
22 00	612	37 00	482	52 00	329	67 00	195
22 30	608	37 30	477	52 30	324	67 30	192
23 00	605	38 00	472	53 00	319	68 00	188
23 30	601	38 30	467	53 30	314	68 30	185
24 00	597	39 00	462	54 00	309	69 00	181
24 30	594	39 30	457	54 30	304	69 30	178
25 00	590	40 00	452	55 00	299	70 00	174
25 30	586	40 30	446	55 30	295	70 30	171
26 00	582	41 00	441	56 00	290	71 00	168
26 30	578	41 30	436	56 30	285	71 30	164
27 00	573	42 00	431	57 00	280	72 00	1·40161
27 30	569	42 30	426	57 30	276		
28 00	565	43 00	421	58 00	271		
28 30	560	43 30	416	58 30	266		
29 00	556	44 00	411	59 00	262		
29 30	552	44 30	406	59 30	257		
30 00	548	45 00	400	60 00	253		
30 30	544	45 30	395	60 30	249		
31 00	539	46 00	390	61 00	244		
31 30	534	46 30	385	61 30	240		
32 00	530	47 00	380	62 00	235		
32 30	1·40525	47 30	1·40375	62 30	1·40231		

The above table is computed for the Clarke spheroid of 1866.

As a check upon the excess computation, the sum of the excesses of triangles that cover the same area must be equal.

647. Number of Angle Equations in a Net. It is evident that each triangle, and also each closed polygon, will give rise to an angle equation. We need only consider lines in our figures, which are observed over in both directions, as those observed in but one direction do not affect the angle equations.

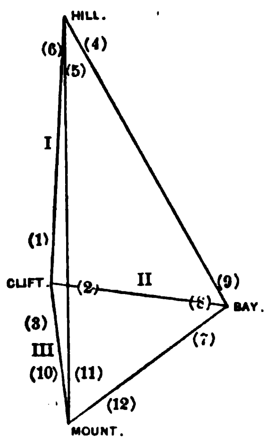
Generally, if s is the number of stations occupied, the polygon forming the outline of the net will give rise to one angle equation. Each diagonal that is drawn will form a figure giving rise to an

additional angle equation. Hence, if in the net there are l lines, observed in both directions, there will be $l - s$ diagonals, and the number of angle equations will be

$$l - s + 1.$$

Example. In Fig. 459 we have a quadrilateral with all lines observed in both directions. Applying the rule for finding the number of angle equations, we have $l = 6$, $s = 4$, whence the number of angle equations necessary to satisfy the angle conditions is $6 - 4 + 1 = 3$. In any complete quadrilateral, therefore, we will require three angle equations.

Fig. 459.



We may form these three equations from any three of the four triangles. We will take Clift—Hill—Bay, Clift—Hill—Mount, and Hill—Bay—Mount. As we determine the corrections for each direction and not the angles, we will have

$$\begin{aligned} -(4)+(5) &= 55^\circ\ 27'\ 42{\cdot}0'' \text{ triangle Hill—Bay—Mount.}\\ -(7)+(9) &= 72\ \ 26\ \ 27{\cdot}1\\ -(11)+(12) &= \underline{52\ \ 05\ \ 53{\cdot}6}\\ &\ 180^\circ\ 00'\ 02{\cdot}7''. \end{aligned}$$

The equation is then

$$0 = +2{\cdot}7 - (4) + (5) - (7) + (9) - (11) + (12), \qquad [1.]$$

where the quantities in parentheses represent the required corrections to the same directions.

$$\begin{aligned} -(1)+(2) &= 87^\circ\ 33'\ 44{\cdot}5'' \text{ triangle Clift—Hill—Bay.}\\ -(4)+(6) &= 59\ \ 25\ \ 32{\cdot}8\\ -(8)+(9) &= \underline{33\ \ 00\ \ 43{\cdot}6}\\ &\ 180^\circ\ 00'\ 00{\cdot}9'' \end{aligned}$$

Equation becomes

$$0 = +0{\cdot}9 - (1) + (2) - (4) + (6) - (8) + (9) \qquad [2.]$$

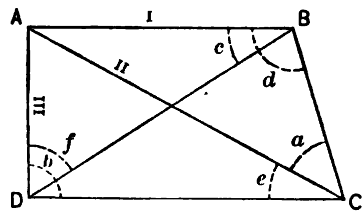
$$\begin{aligned} -(5)+(6) &= 3^\circ\ 57'\ 50{\cdot}8'' \text{ triangle Clift—Hill—Mount.}\\ -(10)+(11) &= 3\ \ 14\ \ 50{\cdot}9\\ -(1)+(3) &= \underline{172\ \ 47\ \ 17{\cdot}2}\\ &\ 179^\circ\ 59'\ 58{\cdot}9''. \end{aligned}$$

Where equation becomes

$$0 = -1{\cdot}1 - (1) + (3) - (5) + (6) - (10) + (11) \qquad [3.]$$

648. Side Equations. In a single triangle or chain of triangles the length of any side can be computed in but one way, hence there can be no side equations. But when the triangles are interlaced this is not the case. In the quadrilateral A B C D, Fig. 460, we may compute II from the base I in two ways, viz.:

Fig. 460.



$$\frac{\mathrm{I}}{\mathrm{II}} = \frac{\sin. a}{\sin. d}, \text{ and from}$$

$$\frac{\mathrm{III}}{\mathrm{I}} = \frac{\sin. c}{\sin. f}$$

$$\frac{\mathrm{II}}{\mathrm{III}} = \frac{\sin. b}{\sin. e}$$

Hence the condition equation or side equation is

$$\frac{\mathrm{I}}{\mathrm{II}}\ \frac{\mathrm{II}}{\mathrm{III}}\ \frac{\mathrm{III}}{\mathrm{I}} = \frac{\sin. a}{\sin. d}\ \frac{\sin. b}{\sin. e}\ \frac{\sin. c}{\sin. f} = 1$$

From this arrangement we see that we may form the side equation in a mechanical manner by assuming one of the stations for a pole (A in this case), and then numbering the lines which radiate from the pole in the order of their azimuths. The equation is then formed from the scheme

$$\frac{\mathrm{I}}{\mathrm{II}}\ \frac{\mathrm{II}}{\mathrm{III}}\ \frac{\mathrm{III}}{\mathrm{IV}} \cdots \frac{n}{\mathrm{I}}$$

(where n is the number of the last line) by replacing I, II, III .... n with the sines of the angles opposite, being careful that the two angles for each fraction, $\frac{\mathrm{I}}{\mathrm{II}}$, $\frac{\mathrm{II}}{\mathrm{III}}$, or $\frac{\mathrm{III}}{\mathrm{IV}}$, etc., are taken from the same triangle.

The pole may be taken at any vertex, but in order to obtain the most perfect adjustment we should have the smallest angles appear in the side equation, hence the pole should be so selected that this will be effected. Whenever a figure has an interior station this

must be taken as the pole. The above method of forming the side equation may be used for any figure.

649. Reduction to Linear Form. In order to carry through the solution by combining the side equations with the other condition equations we must reduce the side equations to the linear form.

If M_1, M_2, etc., represent the measured values of the angles, and $v_1, v_2, \ldots$ the most probable corrections, we have the side equation

$$\frac{\sin.\ (M_1 + v_1)}{\sin.\ (M_2 + v_2)} \quad \frac{\sin.\ (M_3 + v_3)}{\sin.\ (M_4 + v_4)} \cdots\cdots = 1$$

Taking the log. of each side of the equation and expanding by Taylor's theorem, we have, retaining the first powers, only

$$\log.\ \sin.\ M_1 + \frac{d}{d\,M_1}(\log.\ \sin.\ M_1)\, v_1 -$$

$$\left\{\log.\ \sin.\ M_2 + \frac{d}{d\,M_2}(\log.\ \sin.\ M_2) v_2\right\} + \ldots\ldots = 0$$

which may be written in two forms for computation, but we will only use one. If the corrections to the angles are to be expressed in seconds, we may put

$$\frac{d}{d\,M_1}(\log.\ \sin.\ M_1) = \delta'$$

where δ' is the tabulated difference for $1''$ for the angle M_1 in a table of log. sines. Whence

$$\delta' v_1 - \delta'' v_2 + \ldots\ldots + \log.\ \sin.\ M_1 - \log.\ \sin.\ M_2 + \ldots\ldots = 0$$

or,

$$[\delta\, v] = l$$

where l is a known quantity.

It is usually most convenient to use the spherical angles in forming the side equations, but we should get practically the same result by using the plane angles—i. e., the spherical angles corrected for the excess.

Example. In Fig. 459 we have the horizontal directions for each of the lines and wish to form the side equations. The numbers in parentheses are used to represent the various directions; an angle, as Hill—Clift—Bay, is represented by writing $-$ (1) + (2), or Bay—Hill—Clift by $-$ (4) + (6). In this system of notation we merely use the subscripts, and not the quantities themselves, as the work shows whether the angles or their corrections are represented.

Writing the form for the equation, we have, assuming the pole at Clift in order to have the small angles at Hill and Mount enter,

$$1 = \frac{\text{I}}{\text{II}}\;\frac{\text{II}}{\text{III}}\;\frac{\text{III}}{\text{I}} = \frac{\sin.(-(8)+(9))}{\sin.(-(4)+(6))}\;\frac{\sin.(-(10)+(12))}{\sin.(-(7)+(8))}$$
$$\frac{\sin.(-(5)+(6))}{\sin.(-(10)+(11))}$$

For convenience, we write the equation in the following form:

ANGLE.	° ′ ″	LOG. SIN. M.	δ	ANGLE.	° ′ ″	LOG. SIN. M.	δ
$-(8)+(9)$	33 00 43·6	9·73625,01	+0·33	$-(4)+(6)$	59 25 32·8	9·93498,85	+0·18
$-(10)+(12)$	55 20 44·5	9·91518,75	+0·15	$-(7)+(8)$	39 25 43·5	9·80285,45	+0·25
$-(5)+(6)$	3 57 50·8	8·83967,66	+3·04	$-(10)+(11)$	3 14 50·9	8·75319,03	+3·71
		8·49111,42				8·49103,33	
		8·49103,33					
		+8·09	$=l$				

δ and l must be taken in the same unit. In this case the unit is the fifth place in the log. s.

We may now write the side equation in the linear form, taking the δ for each direction, remembering to change the sign for those on the right as they are to be subtracted.

$$0 = +8{\cdot}09 + 0{\cdot}13\,(4) - 3{\cdot}04\,(5) + 2{\cdot}91\,(6) + 0{\cdot}25\,(7) - 0{\cdot}58\,(8)$$
$$+ 0{\cdot}33\,(9) + 3{\cdot}56\,(10) - 3{\cdot}71\,(11) + 0{\cdot}15\,(12)$$

650. Number of Side Equations in a Net. The extremities of the base-line are known, but to fix a third point we must know the other two sides of the triangle, of which this point is the vertex. Hence, if we have a net of triangles connecting s stations, two of the stations being the ends of the base, we must have, in order to plot the figure, $2(s-2)$ lines besides the base, or $2s-3$ lines in all.

Starting from the base, each line in this figure can be computed in but one way; but any additional line, whether observed over in one or both directions, can be computed in two ways, and therefore gives rise to a side equation. If, then, the total number of lines in a net is l_1, observed either in one or both directions, the number of side equations will be indicated by

$$l_1 - 2s + 3.$$

We may also show by similar reasoning that we will satisfy all

the conditions of a figure containing only one base-line by satisfying the angle and side equations determined by the above rules.

In order to guard against omitting any necessary equation, or increasing the work by including one that is unnecessary, it is best in complicated figures to start with a simple figure and form the equations as rapidly as the figure is built up, finally checking by applying the above rules.

Example of Solution. Collecting the angle and side equations for the quadrilateral on page 104, we have the condition equations

$$\left.\begin{array}{l} 0 = +2{\cdot}7 - (4) + (5) - (7) + (9) - (11) + (12) \\ 0 = +0{\cdot}9 - (1) + (2) - (4) + (6) - (8) + (9) \\ 0 = -1{\cdot}1 - (1) + (3) - (5) + (6) - (10) + (11) \end{array}\right\} \text{Angle equations.}$$

$$\left.\begin{array}{l} 0 = +8{\cdot}09 + 0{\cdot}13\,(4) - 3{\cdot}04\,(5) + 2{\cdot}91\,(6) + 0{\cdot}25\,(7) \\ -0{\cdot}58\,(8) + 0{\cdot}33\,(9) + 3{\cdot}56\,(10) - 3{\cdot}71\,(11) + 0{\cdot}15\,(12) \end{array}\right\} \text{Side equations.}$$

Forming the correlate equations by writing the coefficients of the above terms only, we have

CORRELATES.

	C_1.	C_2.	C_3.	C_4.	FINAL RESULTS FOR (1), (2), ETC.
(1)........................	...	−1	−1	......	−0·1
(2)........................	...	+1	...	......	+0·2
(3)........................	...	...	+1	......	−0·1
(4)........................	−1	−1	...	+0 13	+0·3
(5)........................	+1	...	−1	−3·04	+0·1
(6)........................	...	+1	+1	+2·91	−0·5
(7)........................	−1	...	...	+0·25	+0·5
(8)........................	...	−1	...	−0·58	.....
(9)........................	+1	+1	...	+0·33	−0·4
(10)........................	...	...	−1	+3·56	−0·6
(11)........................	−1	...	+1	−3·71	+1·1
(12)........................	+1	...	...	+0·15	−0·5

From the correlate equations we form the normal equations. The first quantity in column 1 is the sum of the squares of the quantities in column C_1 of the correlates; the first in second line of column 2 is the sum of the squares of quantities under C_2; first in line 3 of column 3 is the sum of the squares of quantities in C_3, etc. The upper term of column 2 is obtained by taking the sum of the product of quantities in C_1 and C_2; the top term of 3 by taking the sum of the products of quantities in C_1 and C_3; the term below by taking the sum of the products for C_2 and C_3, etc.

The first column of the normal equations is taken directly from the condition equations.

NORMAL EQUATIONS.

	1	2	3	4	CHECK.
$0=+2\cdot7$	$+6$	$+2$	-2	$+\ 0\cdot77$	$+\ 9\cdot47$
$0=+0\cdot9$		$+6$	$+2$	$+\ 3\cdot69$	$+14\cdot59$
$0=-1\cdot1$			$+6$	$-\ 1\cdot32$	$+\ 3\cdot58$
$0=+8\cdot09$				$+44\cdot6946$	$+55\cdot9246$

Solving these equations either by the direct method, using Crelle's multiplication tables, or by the logarithmic method, or with a machine, we find

$$1 = -0{\cdot}507 \qquad 2 = +0{\cdot}162$$
$$3 = -0{\cdot}081 \qquad 4 = -0{\cdot}188$$

These are really the values for C_1, C_2, C_3, and C_4 respectively. Hence we find the values for (1), (2), etc., in the correlate equations by taking the sum of the products obtained by multiplying the coefficient in each column by the value of the C of that column. We must use all the coefficients on the line with (1) for its value, all on the line with (2) for its value, etc.

The quantities on the right of the correlate equations were obtained in this way, and are the desired corrections to the directions (1), (2), etc., respectively.

No weights were used in this example, but they would only appear in the correlate equations. They must be used as factors for all quantities on the same line when forming the normal equations; and finally, when the corrections for (1), (2), etc., are obtained, they must also be multiplied by the weight of (1), (2), etc., respectively.

We have considered only a quadrilateral here, but it includes all the principles that will be found in more complex figures.

In secondary and tertiary work the triangulation net is broken into simple figures and each adjusted independently. In this case, when a direction is adjusted once it receives no correction if it forms a part of the adjoining figure—i. e., in forming the equations for this second figure consider the correction for this direction zero; hence it does not appear in the equations. In getting

the angles, however, we must use the adjusted value for this direction. In applying the adjustment corrections to the directions we must treat the initial direction just as we do any other; then, after the corrections have all been applied, we may refer all the directions at any one station to the initial station again, by changing all the amount necessary to reduce the initial station to zero.

In primary triangulation a section between two bases is usually adjusted by solving all the equations of the section simultaneously. This, however, causes the solution of so many equations that it is too unwieldy to be used excepting for the highest order of triangulation.

651. Adjustment of the Discrepancy in Bases. Thus far we have considered but one measured base. In an extended triangulation, however, we always have several, hence it becomes necessary to adjust the discrepancy that will arise when we compare the length of one base with its value computed from another base. We could have introduced an equation in the figure adjustment of the same form as the side equation to cover this, but ordinarily we prefer to make the figure adjustment first, as the discrepancy developed affords a good test of the quality of the work. We may then adjust the base discrepancy, through the chain of best shaped triangles, rejecting all tie lines, with as great an accuracy as is necessary.

Rigorous Solution. The condition equation to be satisfied, arising from the connection of the bases, may be written

$$\frac{a+(a)}{b+(b)}=\frac{\sin.\{A_1+(A_1)\}\,\sin.\{A_2+(A_2)\}}{\sin.\{B_1+(B_1)\}\,\sin.\{B_2+(B_2)\}}\cdots\cdot$$

where a, b, A_1, B_1, ... are measured values, and (a), (b), (A_1), (B_1), ... their most probable corrections.

Taking logs. and reducing to the linear form, as on page 106, we have

$$-\delta_a(a)+\delta_b(b)+[\delta_A(A)-\delta_B(B)]=l,$$

where l is excess of the observed over the computed value of log. a, and δ_a, δ_b, δ_A, δ_B, are the logarithmic differences as usual.

Since the angles of each triangle must also satisfy the condition of closure, we have

$$(A_1) + (B_1) + (C_1) = 0$$
$$(A_2) + (B_2) + (C_2) = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

with $\frac{(a)^2}{\mu^2_a} + \frac{(b)^2}{\mu^2_b}\left[\frac{1}{\mu^2}\{(A)^2 + (B)^2 + (C)^2\}\right] =$ a minimum, where μ_a and μ_b are the mean errors of the bases, and $\mu_1, \mu_2, \ldots.$ the mean errors of the angles of each of the triangles.

Example. A simple case is a single triangle with two sides measured.

$$\begin{aligned} & \quad\ \ \text{Feet.} \qquad\quad \text{Feet.} \\ BC &= 6742{\cdot}420 \pm 0{\cdot}010 \\ AC &= 6602{\cdot}386 \pm 0{\cdot}010 \end{aligned}$$

From the figure adjustment, using one base, we get the three angles

$$\begin{aligned} ABC &= \quad\ 1^\circ\ 06'\ 26{\cdot}74'' \pm 0{\cdot}20'' \\ BAC &= \quad\ 1\ \ \ 07\ \ 51{\cdot}35\ \ \pm 0{\cdot}20 \\ ACB &= 177\ \ \ 45\ \ 41{\cdot}91\ \ \pm 0{\cdot}20 \end{aligned}$$

Required the corrections to eliminate the discrepancy between the measured and computed values of either of the measured sides.

The condition equations are

$$(A) + (B) + (C) = 0$$

$$\frac{6742{\cdot}420 + (a)}{6602{\cdot}386 + (b)} = \frac{\sin.\ \{1^\circ\ 07'\ 51{\cdot}35'' + (A)\}}{\sin.\ \{1^\circ\ 06'\ 26{\cdot}74'' + (B)\}}$$

Reducing the latter to the linear form,

$$-\ 0{\cdot}644\ (a) + 0{\cdot}658\ (b) - 1{\cdot}089\ (A) + 1{\cdot}067\ (B) = -\ 0{\cdot}044$$

with $\qquad \frac{(a)^2}{({\cdot}01)^2} + \frac{(b)^2}{({\cdot}01)^2} + \frac{(A)^2}{({\cdot}20)^2} + \frac{(B)^2}{({\cdot}20)^2} + \frac{(C)^2}{({\cdot}20)^2} =$ a min.

The solution of these equations gives

$$\begin{aligned} & \qquad\quad \text{Feet.} \\ (a) &= +\ 0{\cdot}00003 \qquad\qquad & (A) &= -\ 0{\cdot}02'' \\ (b) &= -\ 0{\cdot}00003 \qquad\qquad & (B) &= +\ 0{\cdot}02'' \\ & & (C) &= \quad\ 0{\cdot}00'' \end{aligned}$$

652. Approximate Solution. The rigorous solution is usually too unwieldy for work in general, hence we either adjust (1) the angles alone or (2) the bases alone. As we can get the length of the bases much more accurately than we can make the measurements

of the angles in the triangulation, it is better to correct the angles rather than the bases, although the latter operation is much more simple, and in good work, where the discrepancy is small, may be as accurate as is necessary.

1. When the angles alone are to be changed. The formulas for this case follow from those of the rigorous solution by putting the base corrections equal to zero. If all the adjusted angles are of the same weight the base-line equation becomes

$$[\delta_A (A) - \delta_B (B)] = l,$$

with

$$[(A)^2 + (B)^2 + (C)^2] = \text{a min.}$$

The angle equations are as before.

If k is the correlate of the base-line equation, we have, by eliminating the angle equation correlates,

$$\begin{array}{lll} (A_1) = + (2\,\delta'_A + \delta'_B)\,k & (A_2) = + 2\,\delta''_A + \delta''_B)\;k & \ldots. \\ (B_1) = - (\delta'_A + 2\,\delta'_B)\,k & (B_2) = - (\delta''_A + 2\,\delta''_B)\,k & \ldots. \\ (C_1) = - (\delta'_A - \delta'_B)\;\;k & (C_2) = - (\delta''_A - \delta''_B)\;\;k & \ldots. \end{array}$$

whence, substituting in the base-line equation,

$$k = \frac{l}{2\,[(\delta^2_A + \delta_A)\;(\delta_B + \delta^2_B)]}$$

Hence the corrections to the angles are known. But the corrections to the angles C are small, and vanish entirely when $A = B$, hence, as we have assumed the triangles to be well shaped, we may take

$$(C_1) = (C_2) = \ldots. = 0$$

The angle equation then becomes

$$(A_1) + (B_1) = 0$$
$$(A_2) + (B_2) = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

and

$$(A_1) = - (B_1) = \frac{\delta'_A + \delta'_B}{2}\,k$$
$$(A)_2 = - (B_2) = \frac{\delta''_A + \delta''_B}{2}\,k$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

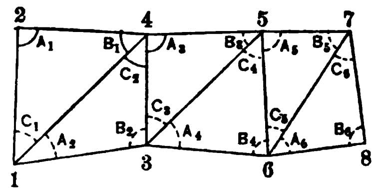
when

$$k = \frac{2\,l}{[(\delta_A + \delta_B)^2]}$$

from which we may obtain the corrections to the angles directly by simple substitution, and do not disturb any of the previous adjustments.

653. Adjustment for Discrepancy in Azimuth. In nearly all triangulation schemes we have several observed azimuths, and also latitudes and longitudes; hence, when making the computation to determine the geodetic positions of the stations, as explained on page 114 *et seq.*, we will develop discrepancies whenever we get a computed value for the stations where such observations were made.

Fig. 461.



The azimuth discrepancy may be adjusted very easily by considering only the chain of best shaped triangles, as in the base-line adjustment. In Fig. 461 we see that 1 — 2, 1 — 4, 4 — 3, etc., are first sides of continuation, and then bases, according to the triangles considered. Hence we may call

A_1, A_2, ... the angles opposite sides of continuation,

B_1, B_2, ... the angles opposite bases,

C_1, C_2, ... the angles opposite flank sides.

The measured azimuths of the lines 1 — 2 and 7 — 8 (a and a_1) are considered correct, and receive no change in the adjustment.

Let the *geodetic* azimuth of 7 — 8 determined from 1 — 2 be represented by a^1, and the difference between the computed and observed by

$$a_1 - a^1 = l_a$$

Reckoning azimuths from the south through west, north, etc., it is easily seen, by passing from 1 — 2 along 1 — 4, 4 — 3, 3 — 5, that l_a is given by

$$+ (C_1) - (C_2) + (C_3) - (C_4) \dots = l_a$$

where (C_1), (C_2), ... denote the corrections to the angles C_1, C_2 .... This is the azimuth condition equation. But we can not disturb the conditions of closure of the triangles, hence we also have the equations

$$(A_1) + (B_1) + (C_1) = 0$$
$$(A_2) + (B_2) + (C_2) = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

The unknowns in these equations are subject to the relation

$$(A_1)^2 + (B_1)^2 + \dots = \text{a min.}$$

Solving by the method of correlates, we obtain

$$(A_1) = \frac{1}{2n} l_a \qquad (A_2) = -\frac{1}{2n} l_a \qquad \dots\dots$$

$$(B_1) = \frac{1}{2n} l_a \qquad (B_2) = -\frac{1}{2n} l_a \qquad \dots\dots$$

$$(C_1) = -\frac{1}{n} l_a \qquad (C_2) = +\frac{1}{n} l_a \qquad \dots\dots$$

when n is the number of triangles passed over.

We see that the C angles get the principal effect of the azimuth adjustment, whereas the A and B angles take most of the base-line adjustment.

654. Computation of Geodetic Positions. Hitherto only the *relative* positions of the various stations have been considered, but in order to orient the whole scheme and place it in its true position upon the surface of the earth, we must determine the absolute position of one or more of the stations, and the direction or azimuth of one or more of the lines of the triangulation scheme. The absolute positions and azimuths are determined by astronomical observations, as explained in Chapter XII.

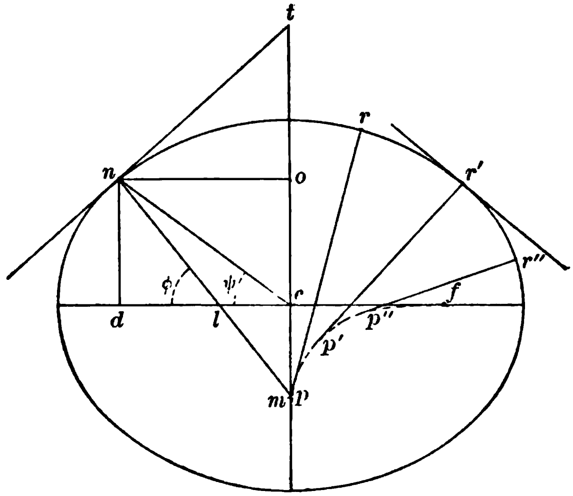
Knowing the absolute position of one station and the azimuth of one of the lines from it, we may compute the geodetic positions of the other stations, using the adjusted values for the angles and sides of the triangulation previously computed. This computation may be effected in two ways: We may either solve the spheroidal triangle formed by the two points and the pole as a whole, arriving at trigonometric functions of the required latitude, azimuth, and difference in longitude, or we may develop expressions from which we can obtain the differences between the given and required quantities, and thus obtain the desired quantities.

The former or direct method requires the use of ten place logarithms, in order to give the positions with a degree of exactness corresponding with the known distance between the two points. The second method contains very convenient expressions for use on the small arcs ordinarily found in most triangulation.

When the arc between the two stations reaches about 2° in length the second method is not sufficiently accurate, and the direct method must be employed. It has been very completely and

elegantly treated by Bessel, and is given in "Astronomische Nachrichten," No. 86, 1826, and will not be considered here, as it is rarely used.

FIG. 462.



The second method was originally developed by Puissant ("Traité de Géodésie," par L. Puissant, troisième edition, tome i, 1842), and the following is but slightly different from his development:

Fig. 462 represents an ellipse, and we have the known relations

$$\epsilon = \frac{a - b}{a} \qquad e^2 = \frac{a^2 - b^2}{a^2}$$

where a is the major or equatorial semi-axis, b the minor or polar semi-axis, ϵ is ellipticity, and e the eccentricity, or distance cf in figure, the distance from the center to the focus.

The latitude, ϕ, of any point is the angle made by the normal of this point with the major axis. The normal, nl, terminating at the major axis, is

$$nl = \frac{a\,(1 - e^2)}{(1 - e^2 \sin.^2 \phi)^{\frac{3}{2}}}$$

When produced to the minor axis,

$$nm = \mathrm{N} = \frac{a}{(1 - e^2 \sin.^2 \phi)^{\frac{1}{2}}}$$

$$cd = no = \mathrm{N} \cos. \phi = \text{radius of a parallel on the spheroid.}$$

Tangent $n\,t$ ending at minor axis $=$ N cot. ϕ.

The ordinate, $$n\,d = \frac{a\,(1 - e^2)\sin.\phi}{(1 - e^2\sin.^2\phi)^{\frac{1}{2}}}$$

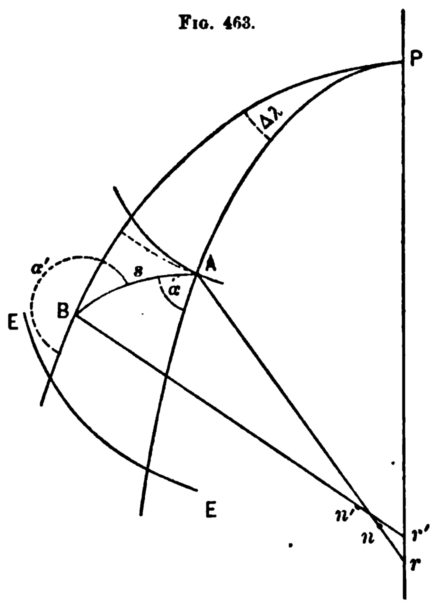
The radius of curvature, R, say $r\,p$, $r'\,p'$, or $r''\,p''$ .... at any point r, r', or r'' .... is

$$R = \frac{a\,(1 - e^2)}{(1 - e^2\sin.^2\phi)^{\frac{3}{2}}}$$

At the equator, sin. $\phi = 0$ and $R = \frac{b^2}{a}$, hence center of curvature is in the focus. At the pole, sin. $\phi = 1$ and $R = \frac{a^2}{b}$.

N and R are the principal quantities used in geodesy. It will be observed that radii of curvature for different latitudes do not intersect unless produced, and then only when they lie in the same meridional plane on the spheroid.

FIG. 463.



A and B in Fig. 463 are two points on a spheroid of revolution, having latitudes ϕ and ϕ', and joined by the geodetic line A B $= s$, making angles with the meridians P A B $= 180° - \alpha$, and P B A $= \alpha' - 180°$. Azimuths are reckoned from south around by west. The angle A P B between the two meridional planes passing through A and B is the difference of their longitudes, λ and λ'. As longitude is reckoned positive to the westward, we have $\lambda' - \lambda = \Delta\,\lambda$. A n and B n' indicate the normals N and N′, and A r and A r' the radii of curvature in the meridian, R and R′ at the points A and B.

Having now the latitude, ϕ, of A given, length s of the geodetic

line A B, and its azimuth, α, at A, we will find ϕ' of B, the angle $\Delta\lambda$, and back azimuth of line A B—i. e., α' at B.

Writing $\gamma = 90° - \phi$, $\gamma' = 90° - \phi'$, $\epsilon = 180° - \alpha$, and σ for the arc A B referred to radius unity, we have in a spherical triangle

$$\cos.\gamma' = \cos.\gamma\cos.\sigma + \sin.\gamma\sin.\sigma\cos.\epsilon.$$

Developing the increment of γ with reference to σ in a rapidly converging series by Taylor's theorem, since σ is never very large, rarely more than 1° or 2°, we have

$$\gamma' = \gamma + \frac{d\gamma}{d\sigma}\sigma + \tfrac{1}{2}\frac{d^2\gamma}{d\sigma^2}\sigma^2 + \tfrac{1}{6}\frac{d^3\gamma}{d\sigma^3}\sigma^3 + \ldots. \qquad [1.]$$

In order to determine the differential coefficients, we consider a differential spherical triangle having the sides γ, $d\sigma$, and $\gamma + d\gamma$, in which

$$\cos.(\gamma + d\gamma) = \cos.\gamma\cos.d\sigma + \sin.\gamma\sin.d\sigma\cos.\epsilon,$$

and developing as usual, we find

$$\frac{d\gamma}{d\sigma} = -\cos.\epsilon,\ \frac{d^2\gamma}{d\sigma^2} = \sin.^2\epsilon\cot.\gamma,\ \frac{d^3\gamma}{d\sigma^3} = \sin.^2\epsilon\cos.\epsilon\,(1 + 3\cot.^2\gamma).$$

Substituting these values in [1], we obtain

$$\gamma' - \gamma = -\sigma\cos.\epsilon + \tfrac{1}{2}\sigma^2\sin.^2\epsilon\cot.\gamma$$
$$+ \tfrac{1}{6}\sigma^3\sin.^2\epsilon\cos.\epsilon\,(1 + 3\cot.^2\gamma) + \ldots.$$

Returning now to ϕ, ϕ', and α, we have

$$\phi - \phi' = \sigma\cos.\alpha + \tfrac{1}{2}\sigma^2\sin.^2\alpha\tan.\phi$$
$$- \tfrac{1}{6}\sigma^3\sin.^2\alpha\cos.\alpha\,(1 + 3\tan.^2\phi) + \ldots. \qquad [2.]$$

If now we refer to an imaginary sphere of radius N with its center at the point where A n intersects the polar axis, we have

$$\sigma = \frac{s}{\mathrm{N}},$$

whence

$$\phi - \phi' = \frac{s\cos.\alpha}{\mathrm{N}} + \tfrac{1}{2}\frac{s^2\sin.^2\alpha\tan.\phi}{\mathrm{N}^2}$$
$$- \tfrac{1}{6}\frac{s^3\sin.^2\alpha\cos.\alpha}{\mathrm{N}^3}(1 + 3\tan.^2\phi) + \ldots. \qquad [3.]$$

But we wish to refer to a sphere of radius R_m, the radius of curvature in the meridian for the middle latitude. As we do not know the middle latitude, it is more convenient to refer to the radius of curvature R of A, whose latitude is known, and afterward determine the small correction to reduce to R_m.

Multiplying equation [3] by $\frac{N}{R}$ and dividing by arc 1″ to express $\phi - \phi' = \delta\phi$, in seconds of arc, we get

$$-\delta\phi = \frac{s}{R \text{ arc } 1''} \cos. a + \tfrac{1}{2} \frac{s^2}{R\, N \text{ arc } 1''} \sin.^2 a \tan. \phi$$

$$- \tfrac{1}{6} \frac{s^3}{R\, N^2 \text{ arc } 1''} \sin.^2 a \cos. a\, (1 + 3 \tan.^2 \phi) + \ldots . \qquad [4.]$$

Tables have been computed giving the logarithms of the following factors for argument ϕ, viz.,

$$B = \frac{1}{R \text{ arc } 1''} \qquad C = \frac{\tan. \phi}{2\, R\, N \text{ arc } 1''}.$$

Substituting in the third term the value of the first $= h$, we can write it

$$\tfrac{1}{6} h \frac{s^2 \sin.^2 a}{N^2} (1 + 3 \tan.^2 \phi)$$

and tabulate another term,

$$E = \frac{1 + 3 \tan.^2 \phi}{6\, N^2}$$

Our formula for computation then becomes

$$-\delta\phi = s \cos. a,\, B + s^2 \sin.^2 a,\, C - h\, s^2 \sin.^2 a,\, E + \ldots . \qquad [5.]$$

We must now determine the correction necessary from the fact that we used R instead of R_m.

If $\Delta\phi$ is the true difference in latitude when R_m is used, then

$$\Delta\phi = \delta\phi \frac{R}{R_m} = \delta\phi \left(1 + \frac{R - R_m}{R_m}\right) = \delta\phi + \delta\phi \frac{R - R_m}{R_m}.$$

Differentiating equation

$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin.^2 \phi)^{\frac{3}{2}}}$$

with respect to R and ϕ, we get

$$d\,R = \frac{a(1 - e^2)(3\, e^2 \sin. \phi \cos. \phi)}{(1 - e^2 \sin.^2 \phi)^{\frac{5}{2}}}\, d\,\phi. \qquad [6.]$$

Since the quantities we are deriving are very small, we may consider them differentials, and taking $d\,\phi$ as the difference in latitude between one extremity of the line s and its middle point—i. e., $d\,\phi = \frac{1}{2}\,\delta\phi$, as given in equation [6], we will obtain the small quantity $d\,R = R - R_m$, whence the correction to $\delta\phi$ becomes

$$\delta\phi \frac{R - R_m}{R_m} = \frac{\frac{3}{2}\, e^2 \sin. \phi \cos. \phi}{1 - e^2 \sin.^2 \phi} (\delta\phi)^2$$

or putting

$$D = \frac{\frac{3}{2} e^2 \sin.\phi \cos.\phi}{1 - e^2 \sin.^2 \phi}$$

we find the next term in formula [6] to be

$$(\delta\phi)^2 D$$

and the complete formula is

$$-\Delta\phi = s\cos.a \,.\, B + s^2 \sin.^2 a \,.\, C + (\delta\phi)^2 D - h s^2 \sin.^2 a \,.\, E. \qquad [7.]$$

This formula admits of a simple and practical computation with the aid of the tabulated logarithmic factors B, C, D, and E.

The last term may be omitted when s is less than about ten statute miles, or log. s in metres less than 4·23; the term $(\delta\phi)^2 D$ may be omitted when log. h is less than 2·31, and h^2 substituted for $(\delta\phi)^2$ when log. s does not exceed 4·93. As previously mentioned, this formula is not sufficiently accurate for long lines, but by taking the fourth differential coefficient neglected in equation [7] we may get a still closer approximation to the true value. This additional term has been developed by Mr. M. H. Doolittle, Computer, United States Coast and Geodetic Survey, in such a way that no further tabulation is necessary. ("Report United States Coast and Geodetic Survey," 1894.)

The additional term of Taylor's theorem is

$$\tfrac{1}{24}\frac{d^4\gamma}{d\sigma^4}\sigma^4 = -\tfrac{1}{24}\sigma^4 \sin.^2\epsilon \cot.\gamma\,[(1 - 3\cos.^2\epsilon)(1 + 3\cot.^2\gamma) - 6\cos.^2\epsilon \operatorname{cosec.}^2\gamma].$$

Substituting as above and multiplying by $\frac{N}{R \operatorname{arc} 1''}$, also as above, we get

$$\begin{aligned}\tfrac{1}{24}\frac{d^4\gamma}{d\sigma^4}\sigma^4 = &-\tfrac{1}{24}\frac{s^4}{R N^3 \operatorname{arc} 1''}\sin.^2 a \tan.\phi\,(1 + 3\tan.^2\phi)\\ &+\tfrac{1}{8}\frac{s^4}{R N^3 \operatorname{arc} 1''}\sin.^2 a \cos.^2 a \tan.\phi\,(1 + 3\tan.^2\phi)\\ &+\tfrac{1}{4}\frac{s^4}{R N^3 \operatorname{arc} 1''}\sin.^2 a \cos.^2 a \tan.\phi \sec.^2\phi.\end{aligned}$$

Denoting the second term of equation [7]

$s^2 \sin.^2 a \,.\, C$ by C_1 we have

$$C_1 E = \frac{s^2 \sin.^2 a \tan.\phi\,(1 + 3\tan.^2\phi)}{12 R N^3 \operatorname{arc} 1''}$$

$$A^2 C_1 = \frac{s^2 \sin.^2 a \tan.\phi}{2 R N^3 \operatorname{arc}^3 1''}$$

Hence we finally obtain

$$-\Delta\phi = s\cos.\alpha\,.\,B + s^2\sin.^2\alpha\,.\,C + (\delta\phi)^2\,.\,D - h\,s^2\sin.^2\alpha\,.\,E$$

$$-\tfrac{1}{6}s^2\,.\,C_1\,E + \tfrac{1}{2}s^2\cos.^2\alpha\,.\,C_1\,E + \tfrac{1}{2}s^2\cos.^2\alpha\sec.^2\phi\,A^2\,C_1\,\text{arc}^2\,1''. \quad [8.]$$

This last part of the equation is always small, and may be safely neglected whenever s is less than about 100 kilometres. It is very easy to introduce this correction, as nearly all the quantities used must be obtained for the first part of the computation.

655. Difference in Longitude. We next deduce the angle A P B (Fig. 463) between the meridional planes passing through A and B and intersecting in the polar axis, or the difference, $\Delta\lambda$, of the longitudes λ and λ' of A and B counted from east to west. We find ϕ' from previous computation, and using the same notation as before, we have

$$\frac{\sin.\gamma'}{\sin.\epsilon} = \frac{\sin.\sigma}{\sin.\Delta\lambda}$$

Referring σ to a sphere whose radius is the normal $B\,n' = N'$, we have $\sigma = \frac{s}{N'}$, and assuming the small arcs, σ and $\Delta\lambda$, proportional to their sines, we have

$$\Delta\lambda = \frac{s\sin.\alpha}{N'\cos.\phi'\,\text{arc}\,1''}$$

We may put $A = \frac{1}{N'\,\text{arc}\,1''}$ and tabulate it as for B, C, D, and E, but it must be taken out for ϕ' instead of ϕ. We then have

$$\Delta\lambda = \frac{s\sin.\alpha}{\cos.\phi'}A \quad [9.]$$

To correct for the assumption that the arcs s and $\Delta\lambda$ are proportional to their sines, we may use a table giving the differences of the logarithms of the arcs and sines. Such a table may be found on page 289, "Coast and Geodetic Survey Report," 1894. Immediately following are the tabulation of the factors A, B, C, D, E, and F.

656. Reverse or Back Azimuth. In the spherical triangle A P B (Fig. 463) we have the relation

$$\cot. \tfrac{1}{2}(\epsilon + \epsilon') = \tan. \tfrac{1}{2}\Delta\lambda \frac{\cos. \frac{1}{2}(\gamma' + \gamma)}{\cos. \frac{1}{2}(\gamma' - \gamma)} = \tan. \tfrac{1}{2}\Delta\lambda \frac{\sin. \frac{1}{2}(\phi' + \phi)}{\cos. \frac{1}{2}(\phi' - \phi)}$$

since $\epsilon = 180° - a$, $\cot. \frac{1}{2}(180° - a + \epsilon') = - \tan. \frac{1}{2}(\epsilon' - a)$

and $$- \tan. \tfrac{1}{2}\Delta a = \tan. \tfrac{1}{2}\Delta\lambda \frac{\sin. \frac{1}{2}(\phi' + \phi)}{\cos. \frac{1}{2}(\phi' - \phi)}$$

Assuming the tangents of Δa and $\Delta\lambda$ proportional to their arcs, and writing ϕ_m for $\frac{1}{2}(\phi' + \phi)$ the middle latitude, we get

$$- \Delta a = \Delta\lambda \frac{\sin. \phi_m}{\cos. \frac{1}{2}\Delta\phi} \text{ and } a' = a + 180° + \Delta a. \qquad [10.]$$

When $\Delta\lambda$ is large we must correct for the assumption that

$$\frac{\tan. \frac{1}{2}\Delta a}{\tan. \frac{1}{2}\Delta\lambda} = \frac{\Delta a}{\Delta\lambda}.$$

The correction is found from

$$\tfrac{1}{12}(\Delta\lambda)^3 \sin. \phi_m \cos.^2 \phi_m \sin.^2 1''$$

and we may tabulate a factor

$$F = \tfrac{1}{12}\sin. \phi_m \cos.^2 \phi_m \sin.^2 1''.$$

Our final formula for determining the reverse azimuth then becomes

$$- \Delta a = \Delta\lambda \frac{\sin. \phi_m}{\cos. \frac{1}{2}(\Delta\phi)} + (\Delta\lambda)^3 F \qquad [11.]$$

The second term is only 0·01″ when $\log. \Delta\lambda = 3·36$.

These formulas may also be used for the solution of the inverse problem—i. e., given ϕ, λ, ϕ', λ', to find s, a, and a'. Put

$$x = s \cos. a = - \frac{1}{B}[\Delta\phi + C \, . \, y^2 + D(\Delta\phi)^2 + E(\Delta\phi)y^2 + E \, . \, C \, . \, y^4]$$

$$y = s \sin. a = \frac{\Delta\lambda \cos. \phi'}{A}$$

whence $$\tan. a = \frac{y}{x}, \; s = x \sec. a = y \operatorname{cosec} a.$$

The following is the form for the direct solution used on the Coast and Geodetic Survey for primary triangulation. For secondary work the terms in parentheses are omitted. For the inverse solution we may use the same form with slight changes.

α	Blue to Pleasant...........................	26°	19′	28·69″
$\angle$	Ragged and Pleasant.........................	−85	35	25·78
α	Blue to Ragged.............................	300	44	02·91
$\Delta\alpha$	....	+	50	03·88
		180		
α'	Ragged to Blue.............................	121°	34′	06·79″

ϕ	44°	43′	41·437″	Blue.	λ	70°	20′	33·157″
$\Delta\phi$	..	30	56·052	$s = 110\,743·7$ metres.	$\Delta\lambda$	−1	11	27·830
ϕ'	44°	12′	45·385″	Ragged.	λ'	69°	09′	05·327″

	FIRST.		SECOND.		THIRD.		FOURTH.
s	5·044 3191	s^2	10·08864			h	3·2633
Cos. α	9·708 4678	sin.^{2} α	9·86854	$(\delta\phi)^2$	6·5372	s^2 sin.^{2} α	9·9572
B	8·510 4887	C	1·39991	D	2·3926	E	6·2069
h	3·263 2756		1·35709		8·9298		9·4274
1st term.	+1833·478″	3d term.	+0·0851″				
2d term.	+ 22·756	[4th term	−0·2675]				
	+1856·234					$(\Delta\lambda)^3$	10·897$_n$
3d [& 4th]						F	7·844
terms.	− 0·182	s					8·741$_n$
$-\Delta\phi$	+1856·052	sin. α	5·044 3191	Arg.		$\Delta\lambda$	
			9·934 2701$_n$				3·632237$_n$
$\frac{1}{2}(\phi+\phi')$	44° 28′ 13·4″	A *	8.509 0107	s	−218	sin. $\frac{1}{2}(\phi+\phi')$	9·845433
$\frac{1}{2}\Delta\phi$	0 15 28·0	sec. ϕ'	0·144 6280	$\Delta\lambda_\mu$	+314	sec. $\frac{1}{2}(\Delta\phi)$	0·000004
			3·632 2375$_n$	Corr.	+ 96		3·477674$_n$
			3·632 2471$_n$			1st term.	−3003·82″
						2d term.	− 0·06
			$\Delta\lambda = -4287·830''$				$-\Delta\alpha = -3003·88$

* A must be taken for ϕ'.

The following constants are based upon the Clark spheroid of 1866 and the metric system, and will enable any one to compute tables similar to those used by the Coast and Geodetic Survey and given in the report for 1894:

$$\log. a = 6·804\,698\,57$$
$$\text{“} \quad b = 6·803\,223\,78$$
$$\text{“} \quad e^2 = \overline{7}·830\,502\,57$$
$$\text{“} \quad \frac{1}{a \text{ arc } 1''} = \overline{8}·509\,726\,56$$

$$\log.\frac{1}{a\,(1-e^2)\,\text{arc}\,1''} = \bar{8}\cdot 512\,676\,15$$

$$\text{“}\quad \frac{1}{2\,a^2\,(1-e^2)\,\text{arc}\,1''} = \bar{1}\cdot 406\,947\,6$$

$$\text{“}\quad (\tfrac{3}{2}\,e^2\,\text{arc}\,1'') = \bar{2}\cdot 692\,168\,7$$

$$\text{“}\quad \frac{1}{6\,a^2} = \bar{5}\cdot 612\,45$$

$$\text{“}\quad (\tfrac{1}{12}\,\text{arc}^2\,1'') = \bar{8}\cdot 291\,96$$

$$\text{A} = \frac{(1-e^2\sin.^2\phi)^{\frac{1}{2}}}{a\,\text{arc}\,1''}$$

$$\text{B} = \frac{(1-e^2\sin.^2\phi)^{\frac{3}{2}}}{a\,(1-e^2)\,\text{arc}\,1''}$$

$$\text{C} = \frac{(1-e^2\sin.^2\phi)^2\tan.\phi}{2\,a^2\,(1-e^2)\,\text{arc}\,1''}$$

$$\text{D} = \frac{\frac{3}{2}\,e^2\sin.\phi\cos.\phi\,\text{arc}\,1''}{1-e^2\sin.^2\phi}$$

$$\text{E} = \frac{(1+3\tan.^2\phi)(1-e^2\sin.^2\phi)}{6\,a^2}$$

$$\text{F} = \tfrac{1}{12}\sin.\phi\cos.^2\phi\,\text{arc}^2\,1''.$$

FIGURE OF THE EARTH.

657. The Shape of the Earth. As the study of geodesy depends primarily upon the shape of the earth, we must discuss more fully what the figure is, and also deduce a number of formulas dependent upon this figure.

The shape of the earth is considered as defined by the sea surface in its mean position regarding tides, etc., which embraces about three fourths of the entire surface. The sea surface * is an equipotential surface, due to the attraction of the earth's mass and to the centrifugal force of its rotation. The position of this surface at any point on the land is defined as the height at which water would stand if a canal connected the point with the ocean.

Various geodetic measurements on the surface of the earth have enabled us to prove that the shape of the earth, as defined by the sea surface, is represented very closely by an oblate spheroid (or ellipsoid of revolution), whose shorter axis coincides with the axis of

* In this discussion, whenever we use the words sea surface we mean the average or mean sea-level surface.

rotation of the earth. This is called the earth's *spheroid.* The actual sea surface, however, is very irregular, owing to varying densities of the masses composing the earth's crust. This actual figure is called the *geoid.* With respect to the spheroid the geoid is a wavy surface lying partly below and partly above; but the extent of the divergence of the two surfaces is probably not more than a few hundred feet at most.

The equation of the generating ellipse of the spheroid with the origin of the axes at the center of the ellipse, and its axes as coordinate axes, is

$$\frac{x^2}{a^2} + \frac{y^2}{b_2} = 1 \qquad [1.]$$

x and y being parallel to the major (a) and minor (b), semiaxes of the ellipse respectively.

We may replace this equation by

$$x = a \cos. \theta \qquad [2.]$$
$$y = b \sin. \theta$$

which give [1] by the elimination of the angle θ. This angle θ is called the *reduced latitude.* (See next paragraph.)

658. Latitudes used in Geodesy. Three different latitudes are used in geodesy, viz.: 1, *astronomical* or *geographical latitude;* 2, *geocentric latitude;* and, 3, *reduced latitude.*

1. The *astronomical* or *geographical latitude* of any place is the angle between the normal (plumb line) at the place and the plane of the earth's equator; or when the plumb line at the place coincides with the normal to the generating ellipse, it is the angle between that normal and the major axis of the ellipse.

2. The *geocentric latitude* of a place is the angle between the equator and a line drawn from the place to the earth's center; or it is the angle between the radius vector of the place and the equator.

3. The *reduced latitude* is defined by equations [2].

The following formulas, and Fig. 462, show the relations between these different latitudes:

ϕ = Astronomical or geographical latitude;

ψ = Geocentric latitude;

θ = Reduced latitude.

From equations [1] and [2] we get

$$\tan.\ \phi = -\frac{d\,x}{d\,y} = \frac{a^2\,y}{b^2\,x}$$

$$\tan.\ \psi = \frac{y}{x}$$

$$\tan.\ \theta = \frac{a\,y}{b\,x}$$

hence $$\tan.\ \psi = \frac{b^2}{a^2}\tan.\ \phi = (1 - e^2)\tan.\ \phi$$

$$\tan.\ \theta = (1 - e^2)^{\frac{1}{2}}\tan.\ \phi = (1 - e^2)^{-\frac{1}{2}}\tan.\ \psi$$

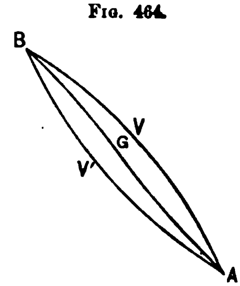
and $$\phi - \psi = \frac{e^2}{2 - e^2}\sin.\ 2\,\phi - \left(\frac{e^2}{2 - e^2}\right)^2\sin.\ 4\,\phi + \ldots.$$

$$\phi - \theta = \frac{a - b}{a + b}\sin.\ 2\,\phi - \tfrac{1}{2}\left(\frac{a - b}{a + b}\right)^2\sin.\ 4\,\phi + \ldots.$$

659. Lines on a Spheroid. The most important lines used in geodesy are *a*, the *curve* of a *vertical section*, *b*, the *geodetic line*, and *c*, the *alinement curve.*

Let A and B, Fig. 464, be two points in the surface of a spheroid. Whenever A and B are not in the same meridian the vertical plane at A containing B will not coincide with the vertical plane at B containing A. The two lines cut from the surface of the spheroid by these two vertical planes are called the *curves of vertical section*, as A V B and A V′ B.

FIG. 464.



The *geodetic line* between A and B is the shortest line between them, and lying in the surface of the spheroid, as A G B, Fig. 464.

The *alinement curve* on a spheroid is a curve whose vertical tangent plane at every point of its length contains the terminal points A and B; or it is a curve containing all the positions which an observer would determine by placing himself in line between A and B such that the telescope of an adjusted transit would point to either A or B by merely transiting it.

The curve *a* lies wholly in one plane, while *b* and *c* are curves of double curvature. In the case of a triangle formed by joining

three points on a spheroid by lines lying in its surface, the curves of class a give two distinct sets of triangle sides, whereas the curves of classes b and c give but one set of sides each.

For all intervisible points on the surface of the earth the lengths of the different lines joining any two points need not be considered, as they are practically equal. The only appreciable differences are in their azimuths. (See below.)

Characteristic Properties of Curves a and b.—a. Curves of Vertical Section.—Let

α_A = azimuth of vertical section at A through B;

α_B = " " " " " B " A;

θ_A, θ_B = reduced latitudes of A and B respectively;

δ_A, δ_B = angles of depression at A and B respectively, of the chord joining the two points.

The characteristic property of the vertical section curves joining A and B is then shown by

$$\sin. \alpha_A \cos. \theta_A \cos. \delta_A = \sin. (\alpha_B - 180°) \cos. \theta_B \cos. \delta_B.$$

b. Of a Geodetic Line.—Let

α'_A = azimuth of the geodetic line at A;

α'_B = " " " " B;

θ_A, θ_B = reduced latitudes of A and B respectively;

θ_0 = reduced latitude of the point where the geodetic line through A and B is at right angles to a meridian plane.

Then the characteristic property of the geodetic line is shown by

$$\sin. \alpha'_A \cos. \theta_A = \sin. (180° - \alpha'_B) \cos. \theta_B = \cos. \theta_0.$$

660. Astronomical and Geodetic Azimuths. The astronomical azimuth is the azimuth of a curve of vertical section, whereas the geodetic azimuth is the azimuth of a geodetic line, hence for refined work we must be able to change from one to the other. This may be done by means of the following formula:

$$\alpha_A - \alpha'_A \text{ (in seconds of arc)} = \frac{1}{12} \frac{e^2 s^2}{a^2 \sin. 1''} \cos.^2 \phi \sin. 2\alpha_A.$$

The notation is the same as above. We may use either α'_A or α_A in the second member of this equation—i. e., either the astronomical or the geodetic, since the quantity we are developing is very small.

$$\log.\left(\frac{1}{12}\frac{e^2}{a^2 \sin. 1''}\right) = 7{\cdot}4244, -20, \text{ for } a \text{ in feet.}$$
$$= 8{\cdot}4564, -20, \text{ for } a \text{ in metres.}$$

Example. $\phi = 0°$, $a_A = 45°$,
$\cos.^2 \phi \sin. 2 a_A = 1$, hence
$$a_A - a'_A = 0{\cdot}074'' \text{ for } s = 100 \text{ miles.}$$
$$= 0{\cdot}296'' \text{ “ } s = 200 \text{ “}$$

Hence this correction is always very small, and need never be considered except in primary work.

661. Determination of the Figure of the Earth. The spheroid which most nearly represents the earth's surface is usually determined by comparing the astronomical and geodetic positions of the same points. If, instead of computing the geodetic positions with reference to an assumed spheroid, we represent the axes of the spheroid by symbols, we may obtain the geodetic positions in terms of these axes. At each point where astronomical observations are made we can form equations between the astronomical and geodetic positions. (If the geodetic positions were computed with reference to the earth's spheroid these equations would really represent the deflection of the plumb line at this point.) From all these equations of condition we can determine the axes of the spheroid, for which the sum of the squares of all the deflections is a minimum. This spheroid will then represent better than any other that part of the earth's surface covered by the triangulation. In order to get the spheroid which best represents the whole earth we must have the points of the triangulation at which astronomical observations have been made well distributed over all the earth's available areas.

The spheroid of reference adopted by most geodesists at present is that determined by Col. A. R. Clarke, R. E., in 1866, known as the Clarke spheroid of 1866. His values for the semiaxes are

$a = 20926062$ feet	$\log. = 7{\cdot}3206875$
$b = 20855121$ “	$\log. = 7{\cdot}3192127$
$e^2 = 0{\cdot}006768658$ “	$\log. = \bar{7}{\cdot}8305030$

In metres we have

$a = 6378206{\cdot}4$ metres	$\log. = 6{\cdot}80469857$
$b = 6356583{\cdot}8$ “	$\log. = 6{\cdot}80322378$

These latter results are based upon Col. Clarke's comparisons between the yard and metre, viz., 1 yard = 0·91439180 metre (or one foot = 0·30479727 metre), but more recent comparisons give the relation, 1 foot = 0·30480061 metre. The difference, however, is very small; hence, as numerous tables have been computed from Col. Clarke's values they are still used, especially as the effect is merely to change slightly the spheroid of reference. With our present knowledge of the figure of the earth one spheroid is as likely to represent it as well as the other.

662. Formulas for the Computation of Arcs of Meridians and Parallels. (*a*) *Arcs of Parallels.*—The radius of any parallel of latitude is equal to the product of the radius of curvature for the normal section for this latitude by the cosine of the latitude—i. e., if r is the radius of the parallel,

$$r = \mathrm{N} \cos. \phi$$

using the notation of page 115. We may then obtain the length Δ P of any arc, $\Delta \lambda$, of a parallel, from

$$\Delta \mathrm{P} = \frac{2 \pi \mathrm{N} \cos. \phi}{360} \Delta \lambda \text{ (for } \Delta \lambda \text{ in degrees)}$$

$$= \frac{2 \pi \mathrm{N} \cos. \phi}{21600} \Delta \lambda \text{ (for } \Delta \lambda \text{ in minutes)}$$

$$= \frac{2 \pi \mathrm{N} \cos. \phi}{1296000} \Delta \lambda \text{ (for } \Delta \lambda \text{ in seconds)}$$

$$\log. \frac{2 \pi}{360} = 8{\cdot}2418774 - 10$$

$$\log. \frac{2 \pi}{21600} = 6{\cdot}4637261 - 10$$

$$\log. \frac{2 \pi}{1296000} = 4{\cdot}6855749 - 10$$

Δ P will be obtained in the same unit as that in which N is taken.

(*b*) *Arcs of Meridians.*—For the precise computation of arcs of the meridian of any length we may use the following formula:

$$\begin{aligned}\Delta \mathrm{M} = \mathrm{A}_0 \Delta \phi &- \mathrm{A}_1 \cos. 2 \phi \sin. \Delta \phi \\ &+ \mathrm{A}_2 \cos. 4 \phi \sin. 2 \Delta \phi \\ &- \mathrm{A}_3 \cos. 6 \phi \sin. 3 \Delta \phi \\ &+ \mathrm{A}_4 \cos. 8 \phi \sin. 4 \Delta \phi \\ &\quad . \quad . \quad . \quad .\end{aligned}$$

where ϕ is the latitude of the middle of the arc or the mean of the latitudes of its extremities, $\Delta\,\phi$ the difference between the latitudes of its extremities, and

$$\begin{aligned}
A_0 &= a\,(1+n)^{-1}\,(1+\tfrac{1}{4}\,n^2+\tfrac{1}{64}\,n^4+\ldots.)\\
A_1 &= 3\,a\,(1+n)^{-1}\,(n-\tfrac{1}{8}\,n^3-\ldots.)\\
A_2 &= \tfrac{15}{8}\,(1+n)^{-1}\,(n^2-\tfrac{1}{4}\,n^4-\ldots.)\\
A_3 &= \tfrac{35}{24}\,(1+n)^{-1}\,(n^3-\ldots.)\\
A_4 &= \tfrac{315}{256}\,(1+n)^{-1}\,(n^4-\ldots.)
\end{aligned}$$

where $n = \dfrac{a-b}{a+b}$

Adopting the Clarke spheroid of reference, we have

	Feet.	Log. in feet.	Log. in metres.
$A_0 =$	20890606	7·3199510	6·8039668
$A_1 =$	106411	5·0269880	4·5110038
$A_2 =$	113	2·0528	1·5368
$A_3 =$	0·15	9·174 − 10	8·658 − 10

A_4 is not necessary, as it is considerably smaller than the uncertainty of A_0.

If we express $\Delta\,\phi$ in degrees, the above equation for the Clarke spheroid of 1866 becomes

$$\begin{aligned}
\Delta\,M = {} & [5{\cdot}5618284]\ \Delta\,\phi \text{ (for } \Delta\,\phi \text{ in degrees)}\\
& - [5{\cdot}0269880]\ \cos.\ 2\,\phi\ \sin.\ \Delta\,\phi\\
& + [2{\cdot}0528]\quad \cos.\ 4\,\phi\ \sin.\ 2\,\Delta\,\phi\\
& \ldots\ldots
\end{aligned}$$

the quantities in brackets being logarithms. The result will be given in feet. To obtain Δ M in metres, add the log. 9·4840158 to each of the above logarithms.

Very complete tables giving the lengths of arcs of the meridian and parallels may be found in "United States Coast and Geodetic Survey Report" for 1884, Appendix 6. They give the length of each second and minute of arc, to the argument ϕ for each minute of latitude from the equator to the pole, and are based on the Clarke's spheroid of 1866.

The following formulas were used for the computation:

$$\Delta\,M_1 = 111132{\cdot}09 \text{ m.} - 566{\cdot}05 \text{ m. } \cos.\ 2\,\phi + 1{\cdot}20 \text{ m. } \cos.\ 4\,\phi - 0{\cdot}003 \text{ m. } \cos.\ 6\,\phi$$

$$\Delta\,P_1 = 111415{\cdot}10 \text{ m. } \cos.\ \phi - 94{\cdot}54 \text{ m. } \cos.\ 3\,\phi + 0{\cdot}12 \text{ m. } \cos.\ 5\,\phi$$

when $\Delta\,M_1$ and $\Delta\,P_1$ are the lengths in metres of one degree.

These tables also contain the co-ordinates of curvature for the polyconic projection of maps, for differences of longitude from one minute (1′) to 30°, and tabulated for each degree of latitude.

663. Map-making. As the surface of a sphere or a spheroid can not be developed, the problem of representing any large portion of the earth's surface on a plane presents considerable difficulties. Many methods have been employed in map-making, and the one to be chosen in any given case depends upon the purpose for which the map is to be used. All the methods of map-making may be classed in three groups: The method may be purely conventional; or it may be a projection of a portion of the earth on a plane; or the projection may be made on a developable surface, such as an enveloping cylinder or cone, and then this enveloping surface may be developed.

As points on the earth's surface are usually given by spherical co-ordinates, such as latitude and longitude, or azimuth and distance, it is necessary that the lines on which these co-ordinates are to be measured be represented by some method so that points may be mapped by their co-ordinates, and so that, when the map is made, the co-ordinates of points may be determined from the map. The first thing, then, to be considered is the method of representing the meridians and parallels of latitude.

Conventional Methods.—One of the simplest of the conventional methods is to conceive the plane of the map to coincide with a meridian. Then, a straight line drawn from the north pole to the south pole will be the central meridian of the hemisphere to be represented, and a perpendicular at its middle point will be the equator. Then divide the equator into twelve equal parts, and pass arcs of circles through the points of division and the north and south poles. These arcs will represent the meridians. Divide each of the four quadrants from the poles to the equator into nine equal parts, and the central line, representing the central meridian, into eighteen equal parts, and pass arcs of circles through the points of division on the central meridian and the corresponding points on the quadrants on both sides. These arcs will represent the parallels

of latitude. This is called the *equidistant projection*, but it is not a projection.

Lorgna's Map.—In this method a circle is drawn having a radius equal to R $\sqrt{2}$, R being the radius of the sphere. The area of this circle will be equal to the surface of the hemisphere. Through the center of the circle draw straight lines making angles of 15° with each other. These lines will represent meridians, the center of the circle being the pole. Any parallel is drawn by describing a circle with the pole as a center and a radius equal to $\sqrt{2 R h}$, h being the altitude of the zone included between the pole and the plane of the parallel.

The area of this circle will be equal to the area of the zone between the pole and the parallel. The area of any quadrilateral, bounded by arcs of the circles representing parallels and the straight lines representing meridians, will be equal to the surface which it represents on the hemisphere.

The Square Projection.—In this the meridians and parallels are straight, equidistant lines, forming squares. The convergence of the meridians is ignored, and the degrees of latitude and longitude are supposed to be of equal length. The farther the map extends from the equator the greater will be the distortion.

The Rectangular Equal Surface Projection.—In this the above method is modified by drawing the parallels at a distance from the equator proportional to the sine of the latitude.

The Method of Converging Meridians.—In this a straight line is drawn representing the central meridian of the map. On this line is laid off degrees, or multiples or parts of degrees, to the scale of the map. Through these points of division lines representing parallels are drawn perpendicular to the central meridian. On the top and bottom parallel are laid off to scale the degrees, or multiples or parts of degrees, of the proper length belonging to the latitude which each parallel represents. Corresponding points of division on the top and bottom parallel, at the same angular distance from the central parallel, are connected by straight lines which represent the meridians. In this method only the central meridian is at right angles to the parallels, and only the upper and lower parallels give degrees of longitude in their true proportions.

Flamsteed's Method.—Draw a straight line to represent the central meridian of the region to be mapped. On this lay off degrees, or multiples of degrees, to scale. Through these points of division, and perpendicular to the central meridian, draw straight lines to represent the parallels of latitude. These parallels are then divided into degrees, or multiples of degrees, to scale, as they would be on the sphere, and curves are passed through the points of division, connecting the points of equal longitude. These curves represent the meridians. The principal objection to this method is that the meridians are not perpendicular to the parallels.

Spherical Projections.—Let the plane upon which the projection is to be made pass through a great circle of the sphere. This plane is called the primitive plane, and the great circle coinciding with this plane the primitive circle. If a projection be made with the point of sight in the axis of the primitive circle, and at an infinite distance from the plane, it is called an *orthographic* projection.

If the point of sight be taken in the axis of the primitive circle, and outside the surface of the sphere a distance of $R\sqrt{\frac{1}{2}}$, the projection is called *globular*.

If the point of sight be taken at the intersection of the axis of the primitive circle with the surface of the sphere, the projection is called *stereographic*.

If the plane on which the projection is made be tangent to the sphere, and the point of sight be taken at the center, the projection is called *gnomonic*.

On spherical projections, see Church's "Descriptive Geometry," Part II.

These projections are usually confined to maps representing a hemisphere, and are rarely used for maps of limited areas on a large scale, as are needed in geodetic surveying.

Cylindrical Projection.—If a cylinder be passed tangent to the equator, the circles of the sphere projected on the cylinder, and the cylinder developed, the meridians and parallels will be developed into straight lines, the meridians being at right angles to the parallels. This is the cylindrical projection. There are several modifications of this projection, as to position of the cylinder and

of the point of sight. The cylinder may be passed through the middle parallel of the area to be mapped, the point of sight may be taken at the center of the sphere, or the projecting lines may be perpendicular to the elements of the cylinder. In some of these modifications of cylindrical projection the position of the parallels is not found by projection, but is determined by some assumed law.

Mercator's Projection is a modification of the cylindrical. The cylinder is conceived to be tangent at the equator. Then the equator will be developed into a right line, and the meridians into right lines perpendicular to the equator. The parallels of latitude are straight lines parallel to the equator, and are so drawn that the representative of a degree of latitude at any distance from the equator is increased in the same ratio as the representative of a degree of longitude has been increased by making the meridians parallel.

The length of a degree of longitude at any latitude is equal to the length of a degree at the equator, multiplied by the cosine of the latitude.

By making the meridians parallel, the representative of a degree of longitude, compared with the arc itself, is increased as the cosine of the latitude decreases or as the secant increases. Hence, the representative of a degree of latitude should be increased in the same ratio. By beginning at the equator and adding the parallels so that each degree of latitude, or, still more accurately, each small part of a degree, shall be given the above value, the relative ratio of the representatives of distance in latitude and longitude will be preserved. Charts made in this way are much used for navigation. A line on one of them having the same bearing throughout—that is, meeting the meridians at the same angle—is a straight line. The parts in high latitudes are very much enlarged.

Conical Projection.—If a cone be passed tangent to a sphere on one of its parallels of latitude and the circles of the sphere be projected on the cone, the point of sight being at the center of the sphere, the meridians will be projected into right lines coinciding with the elements of the cone, and the parallels into circles parallel to the line of tangency. When the cone is developed, the meridians will be straight lines passing through the vertex, and each parallel

will be an arc of a circle with the vertex as a center, and a radius equal to its distance from the vertex. This is the conical projection.

To lessen the exaggeration in the parts distant from the line of contact, the cone may be passed through two parallels, one of the parallels to be one half of the distance from the center of the region to be mapped to its northern boundary, and the other parallel one half of the distance from the center to the southern boundary.

De Lisle's Projection is a modification of the conic projection. In this the central meridian of the region to be mapped is first drawn. Through the middle point of this meridian an arc of a circle is drawn, whose center is on the central meridian produced, whose radius is equal to the cotangent of the latitude, and whose length is sufficient to include the number of degrees of longitude desired in the map. As many equidistant points of division are then marked on the central meridian as parallels of latitude are required on the map. Through these points of division arcs of circles are drawn to represent the parallels of latitude. The common center of these arcs is the center of the arc passed through the middle point of the central meridian.

On two of the parallels, one taken at one fourth and the other at three fourths of the height of the map, degrees or multiples of degrees are laid off to scale. Through these points of division on the parallels, representing the same degree of longitude, straight lines are drawn. These straight lines represent the meridians. This map will give the true lengths of arcs on the central meridian, and on the two parallels which were graduated.

Bonne's Projection.—This also is a modification of the conic. Draw a straight line to represent the central meridian of the region to be mapped, and divide it into degrees, or multiples or subdivisions of a degree. Through the middle point of a division draw an arc of a circle whose center is on the central meridian produced, and whose radius is equal to the cotangent of the latitude, as in the conic projection.

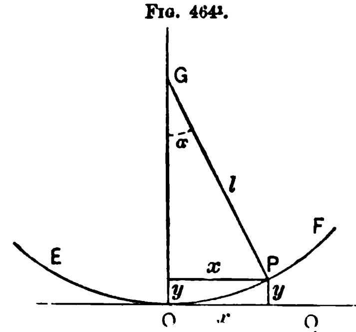
Through the points of division of the central meridian pass arcs of circles whose common center is the center of the middle parallel. These arcs will represent the parallels of latitude. On these paral-

lels lay off the degrees, or subdivisions or multiples of degrees, as they are on the sphere. Through these points of division on the parallels pass curved lines, each line passing through points having the same angular distance from the central meridian. These curved lines will represent the meridians.

Polyconic Projection.—In this projection the central meridian and central parallel of latitude are drawn as in the conic projection. Then another cone is conceived to be passed tangent to the next parallel on the map, and this parallel is developed as before. This process is continued for all the parallels desired in the map. The radius of each parallel will be equal to the distance from the tangent circle to the vertex of the cone—that is, equal to the cotangent of the latitude. Each parallel is then divided into degrees or fractions of a degree, and lines are drawn through the points of division on the parallels having the same longitude. These lines represent the meridians.

In this projection the lines representing parallels of latitude are not quite parallel, and all the meridians are not precisely perpendicular to the parallels. Neither of these distortions, however, are excessive, even for a whole hemisphere, and for such maps and charts as would be needed in a geodetic survey the distortion can scarcely be detected.

FIG. 464[1].



In Fig. 464[1] let O G represent the central meridian, E F the developed arc of the parallel, P G being its radius, or the slant height of the tangent cone for this parallel. Any point of this arc may be represented by the co-ordinates x and y, as shown in the figure, O Q being perpendicular to O G, and x and y parallel to them respectively. To find x and y, we have

$$x = l \sin. a$$
$$y = 2\, l \sin.^2 \tfrac{1}{2}\, a$$

a being the angle at G subtended by the arc O P, and l the radius of the developed parallel.

But $\quad l = \text{N cot. } \phi \text{ and } a = \Delta \lambda \text{ sin. } \phi$

whence $\quad x = \text{N cot. } \phi \text{ sin. } (\Delta \lambda \text{ sin. } \phi)$

$$y = 2 \text{ N cot. } \phi \text{ sin.}^2 \tfrac{1}{2} (\Delta \lambda \text{ sin. } \phi)$$

Any desired ordinates may be computed by these formulas. They were used in the computation of the extensive tables given in the "United States Coast and Geodetic Survey Report," 1884, Appendix 6.

The Equidistant Polyconic Projection.—This is a modification of the polyconic method. The meridians and parallels are first determined by the polyconic method, the parallels, excepting the middle one, being drawn in pencil. Then, to determine the new positions for the parallels, the distance on the central meridian between the central parallel and the consecutive parallels is laid off along the meridians, north and south of the central parallel to the limit of the map, and through those points of division the parallels are passed. The first parallels, drawn in pencil, are then erased. This method brings the parallels of latitude parallel on the map, and avoids the slight distortion in latitude of the regular polyconic method.

References.—"A Treatise on Projections," by Thomas Craig, United States Treasury Department, Document No. 61; "A Comparison of the Polyconic with other Projections," by Charles A. Schott, in "Coast and Geodetic Survey Report" for 1880, Appendix 15.

CHAPTER XII.

FIELD ASTRONOMY.

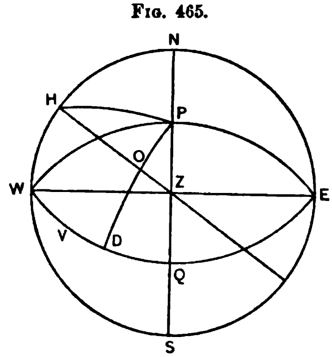
664. THE celestial sphere is a sphere to which it is convenient to refer stars and other celestial objects. Its center is assumed to be coincident with the eye of the observer, and the objects referred to it are supposed to lie in its surface. The orientation of this sphere is defined by its equator, which is assumed to be parallel to the earth's equator. The plane of the equator is thus the principal plane of reference. Other planes of reference are the plane of the *horizon*, which is at right angles to the plumb line at the place; the *meridian*, which is a plane through the place and the earth's axis of rotation; the *prime vertical*, which is a vertical plane at the place at right angles to the meridian; and the *ecliptic*, which is a plane parallel to the plane of the earth's orbit. These planes cut the surface of the sphere in great circles called the equator, the horizon, the meridian, etc. The points on the sphere defined by the intersection of the meridians, or the points where the axis of the equator pierces the sphere, are called the *poles*. Similarly, the prolongation of the plumb line upward pierces the sphere in the *zenith*, and the prolongation downward pierces the sphere in the *nadir*. Great circles passing through the zenith are called *vertical circles*.

The *obliquity of the ecliptic* is the angle which it makes with the equator. The points where the ecliptic and equator intersect are called the *equatorial points* or *equinoxes;* and that diameter of the celestial sphere in which their planes intersect is the *line of equinoxes*. The *vernal equinox* is the one where the sun ascends from the southern to the northern side of the equator, and the *autumnal equinox* the one where it descends from the northern to the southern side of the equator. The *solstitial points*, or *solstices*, are the points of the ecliptic 90° from the equinoxes. They are

distinguished as northern and southern, or summer and winter solstices.

665. Spherical Co-ordinates. The position of a celestial body may be defined by several systems of co-ordinates. The most important of these in practical astronomy are the *azimuth* and *altitude* system, the *hour-angle* and *declination* system, and the *right-ascension* and *declination* system. In the first system, the *azimuth* of a star or other body is the angle between the meridian plane of the place and a vertical plane through the star. It is measured, in general, from the south around by the west through 360°. The *altitude* of a star is its angular distance above the horizon, and its zenith distance is the complement of the altitude. In the second system, the *hour angle* of a star is the angle between the meridian plane of the place and a meridian plane through the star. It is measured toward the west through 360°. The *declination* of a star is its angular distance above or below the equator; the complement of the declination is called the *polar distance.* In the third system, the *right ascensicn* of a point of the sphere is the arc of the equator intercepted between its meridian, or circle of declination, and the vernal equinox, and is reckoned from the vernal equinox eastward from 0° to 360° or 0^h to 24^h. Right ascension and declination are not affected by the diurnal motion of the earth, hence their values vary only with the time, and are given in the ephemerides as functions of the time reckoned at some assumed meridian.

Fig. 465.



The angular distance of the pole above the horizon is equal to the zenith distance of the equator, or to the latitude of the place. Likewise the altitude of the equator and the zenith distance of the pole are each equal to the complement of the latitude at any place. These quantities are usually designated by the following notation, and are shown in Fig. 465, which represents the stereographic pro-

jection of the celestial sphere upon the plane of the horizon, the projecting point being the nadir. The equator, W Q E, passes through the east and west points of the horizon, and all vertical circles passing through the projecting point are projected into straight lines, as the meridian N Z S, the prime vertical W Z E, and the vertical circle Z O H through any point O of the surface of the sphere.

A_n = the azimuth of a star or object O, Fig. 465, from the north, = N E S H, or angle N Z H;

h = its altitude, corrected for refraction, parallax, etc., or its true altitude, = O H;

z = its zenith distance = $90° - h$ = O Z;

t = its hour angle, = Q D or angle Q P D;

δ = its declination, = O D;

α = right ascension, = V D or angle V P D, V being the vernal equinox;

p = its polar distance, = $90° - \delta$ = O P;

q = the parallactic angle, or angle at the star between the pole and the zenith;

ϕ = the geographical latitude of the place of observation, = Z Q = $90°$ − P Z = P N.

666. Altitude and Azimuth in Terms of Declination and Hour Angle. Applying the principles of spherical trigonometry (see Fig. 465), the fundamental relations for this problem are

$$\begin{aligned} \sin. h &= \sin. \phi \sin. \delta + \cos. \phi \cos. \delta \cos. t \\ \cos. h \cos. A_n &= -\cos. \phi \sin. \delta + \sin. \phi \cos. \delta \cos. t \\ \cos. h \sin. A_n &= \cos. \delta \sin. t \end{aligned} \qquad [1.]$$

When it is desired to compute both A_n and h by means of logarithms, the most convenient formulas are

$$m \sin. M = \sin. \delta, \qquad m \cos. M = \cos. \delta \cos. t, \qquad \tan. M = \frac{\tan. \delta}{\cos. t}$$

$$\sin. h = m \cos. (\phi - M), \quad \tan. A_n = \frac{\tan. t \cos. M}{\sin. (\phi - M)} \qquad [2.]$$

$$\cos. h \cos. A_n = m \sin. (\phi - M), \qquad \cos. h \sin. A_n = \cos. \delta \sin. t, \qquad \tan. h = \frac{\cos. A_n}{\tan. (\phi - M)}$$

$A_n > 180°$ when $t > 180°$, and $A_n < 180°$ when $t < 180°$.

For the computation of A_n and z separately, the following formulas are useful:

$$\tan. A_n = -\frac{\sin. t}{\cos. \phi \tan. \delta (1 - \tan. \phi \cot. \delta \cos. t)}$$

$$= -\frac{a \sin. t}{1 - b \cos. t} \qquad [3.]$$

where $\quad a = \sec. \phi \cot. \delta, \qquad b = \tan. \phi \cot. \delta$

667. Declination and Hour Angle in Terms of Altitude and Azimuth. The fundamental relations for this case are

$$\begin{aligned} \sin. \delta &= \sin. \phi \sin. h - \cos. \phi \cos. h \cos. A_n \\ \cos. \delta \cos. t &= \cos. \phi \sin. h + \sin. \phi \cos. h \cos. A_n \\ \cos. \delta \sin. t &= \qquad\qquad \cos. h \sin. A_n \end{aligned} \qquad [4.]$$

For logarithmic computation by means of an auxiliary angle, M, one may write

$$\begin{aligned} m \sin. M &= \cos. h \cos. A_n & \tan. M &= \cot. h \cos. A_n \\ m \cos. M &= \sin. h & \tan. t &= \frac{\tan. A_n \sin. M}{\cos. (\phi - M)} \\ \sin. \delta &= m \sin. (\phi - M) \\ \cos. \delta \cos. t &= m \cos. (\phi - M) \\ \cos. \delta \sin. t &= \cos. h \sin. A_n, \quad \text{and} & \tan. \delta &= \tan. (\phi - M) \cos. t \end{aligned} \qquad [5.]$$

668. Hour Angle and Azimuth in Terms of Zenith Distance.

$$\cos. t = \frac{\cos. z - \sin. \phi \sin. \delta}{\cos. \phi \cos. \delta} \qquad [6.]$$

$$\tan.^2 \tfrac{1}{2} t = \frac{\sin. (s - \phi) \cos. (s - \delta)}{\cos. s \cos. (s - z)}, \qquad s = \tfrac{1}{2}(\phi + \delta + z) \qquad [7.]$$

$$\cos. A_n = \frac{\sin. \phi \cos. z - \sin. \delta}{\cos. \phi \sin. z} \qquad [8.]$$

$$\tan.^2 \tfrac{1}{2} A_n = \frac{\sin. (s - \phi) \cos. (s - z)}{\cos. s \sin. (s - \delta)}, \qquad s = \tfrac{1}{2}(\phi + \delta + z) \qquad [9.]$$

669. Formulas for Parallactic Angle.

$$\begin{aligned} \cos. z &= \sin. \delta \sin. \phi + \cos. \delta \cos. \phi \cos. t \\ \sin. z \cos. q &= \cos. \delta \sin. \phi - \sin. \delta \cos. \phi \cos. t \\ \sin. z \sin. q &= \qquad\qquad \cos. \phi \sin. t \\ \sin. \delta &= \cos. z \sin. \phi + \sin. z \cos. \phi \cos. t \\ \cos. \delta \cos. q &= \sin. z \sin. \phi + \cos. z \cos. \phi \cos. A_n \\ \cos. \delta \sin. q &= \qquad\qquad \cos. \phi \sin. A_n \end{aligned} \qquad [10.]$$

The first three of these are adapted to logarithmic computation, as follows:

$$\begin{aligned} n \sin. \mathrm{N} &= \cos. \phi \cos. t \\ n \cos. \mathrm{N} &= \sin. \phi \\ \cos. z &= n \sin. (\delta + \mathrm{N}) \\ \sin. z \cos. q &= n \cos. (\delta + \mathrm{N}) \\ \sin. z \sin. q &= \cos. \phi \sin. t \end{aligned} \qquad [11.]$$

whence

$$\begin{aligned} \tan. \mathrm{N} &= \cot. \phi \cos. t \\ \tan. z \sin. q &= \frac{\tan. t \sin. \mathrm{N}}{\sin. (\delta + \mathrm{N})} \\ \tan. z \cos. q &= \cot. (\delta + \mathrm{N}) \end{aligned} \qquad [12.]$$

A similar adaptation results for the last three of equations [10] by interchanging δ and z. The equations [12] give both z and q in terms of ϕ, δ, and t without ambiguity, since tan. z is positive for stars above the horizon.

If A_n, z, and q are all required from ϕ, δ, and t, they are best given by the Gaussian relation:

$$\begin{aligned} \sin. \tfrac{1}{2} z \sin. \tfrac{1}{2} (\mathrm{A} + q) &= \sin. \tfrac{1}{2} t \cos. \tfrac{1}{2} (\phi + \delta) \\ \sin. \tfrac{1}{2} z \cos. \tfrac{1}{2} (\mathrm{A} + q) &= \cos. \tfrac{1}{2} t \sin. \tfrac{1}{2} (\phi - \delta) \\ \cos. \tfrac{1}{2} z \sin. \tfrac{1}{2} (\mathrm{A} - q) &= \sin. \tfrac{1}{2} t \sin. \tfrac{1}{2} (\phi + \delta) \\ \cos. \tfrac{1}{2} z \cos. \tfrac{1}{2} (\mathrm{A} - q) &= \cos. \tfrac{1}{2} t \cos. \tfrac{1}{2} (\phi - \delta) \end{aligned} \qquad [13.]$$

670. Hour-Angle, Azimuth, and Zenith Distance of a Star at Elongation. In this case the parallactic angle is 90°, and the required quantities are given by the formulas

$$\begin{aligned} \cos. t &= \frac{\tan. \phi}{\tan. \delta} \\ \sin. \mathrm{A} &= \frac{\cos. \delta}{\cos. \phi} \\ \cos. z &= \frac{\sin. \phi}{\sin. \delta} \end{aligned} \qquad [14.]$$

671. Star Places. In order to determine our position on the surface of the earth by observations upon the stars, or sun, we must be able to obtain the position that the body occupies at the instant of the observation, or, in other words, its co-ordinates with reference to two fixed planes. These co-ordinates are usually *right ascension* and *declination*.

In the case of the sun, moon, or planets, we must depend upon the data given in an ephemeris, such as the "American Ephemeris," "Nautical Almanac" (English), "Berliner Jahrbuch," etc., as the prediction of their places for any given instant is very intricate. With the fixed stars, however, the case is different. Their relative positions change very slightly, and we have merely to consider, practically, the motion of the earth and planes of reference. The motion that a fixed star actually has is called its *proper motion*. It is always a very small quantity, and is considered uniform for all ordinary astronomical work. It rarely exceeds 1·5″ per annum in declination for the so-called fixed stars, and is less than 0·1″ per annum for most of them.

The apparent co-ordinates of all fixed stars are varying slowly, but continuously, owing to two causes which are independent of the star's motion—viz., first, a shifting of the planes of reference, causing what is known as *precession* and *nutation* and, second, an apparent motion of the star due to the earth's motion combined with the progressive motion of light, which is called *aberration*.

672. Precession is the name applied to the slow motion of the vernal equinox, thus causing the pole of the equator to describe a circle about the pole of the ecliptic in a period of about twenty-five thousand years. This motion is due to the spheroidal shape of the earth, in consequence of which the attraction of the sun and moon tends to draw the equator into coincidence with the ecliptic. This force is not uniform, but is a maximum when the sun and moon are farthest from the equator, and a minimum when they are in the equator.

673. Nutation is the name applied to all the small changes of short period caused by the want of uniformity in the forces producing the precession. The principal one of these small changes causes the actual pole of the earth's equator to describe a small ellipse about the mean pole. The axes of this ellipse are about 18″ and 14″, the major being directed toward the pole of the ecliptic; the period is eighteen or nineteen years.

674. The Aberration of the Fixed Stars. The only one we need to consider is determined by the velocity and direction of motion of the point on the earth's surface occupied by the observer. There are three of these motions—viz., first, that due to the diurnal revolution of the earth on its axis (which is only considered in connection with time and azimuth work); second, that due to the earth's annual motion about the sun; and, third, that due to the motion of the earth with the sun in space. This last motion need not be considered, as it affects the position of the star by a constant quantity, and is not well known.

The annual aberration produced by the earth's motion in its orbit is therefore the only one considered in connection with the determination of the apparent star places.

675. By **Apparent Place**, as applied to a celestial body, is meant its position as actually observed—i. e., the observed right ascension and declination. If the correction for aberration be applied to the apparent place, we obtain the *true place*; and by also applying the corrections for nutation, we obtain the *mean place.*

Mean Places.—In all star catalogues the mean places can be given for a certain epoch only, hence there are also given the annual precession and proper motion, and the *secular variation* in the precession (i. e., its change in one hundred years), so as to enable the determination of the mean place for the date required.

If t is the number of years between the epochs of the catalogue and the observations,

p the annual precession for the epoch of the catalogue,

Δp the secular variation,

μ the proper motion (always annual),

we have the correction, c, to the catalogue mean place necessary to obtain the mean place for the year of the observation from

$$c = \left(p + \frac{\Delta p}{200}\, t + \mu\right) t$$

for either right ascension or declination.

The precession is determined for the middle of the interval, t.

As all catalogues give the mean place for the beginning of the

year, the mean place we have obtained will be that for the beginning of the year in which the observations were made.

Apparent Places.—Having the mean places for the beginning of the year, we have to determine the corrections for precession and nutation necessary to obtain the true places at the instant of the observation, and then the corrections for aberration to get the apparent places.

The theory and methods of obtaining these corrections may be found in works on practical astronomy. It is not necessary to reproduce them here. The formulas thus obtained are to be used in connection with an ephemeris in which are given the constants of the formulas, and the star numbers dependent upon these constants. On page 280 of the "American Ephemeris" for 1882–1900 may be found the constants and the formulas for obtaining the factors used in the reduction of the apparent places of the fixed stars. When a great many determinations are desired for the same star it is more convenient to use the Besselian star numbers and the corresponding formulas—viz.:

$$\alpha = \alpha_0 + \tau\,\mu + \mathrm{A}\,a + \mathrm{B}\,b + \mathrm{C}\,c + \mathrm{D}\,d + \tfrac{1}{15}\,\mathrm{E} \qquad \text{(in time).} \quad [15.]$$

$$\delta = \delta_0 + \tau\,\mu' + \mathrm{A}\,a' + \mathrm{B}\,b' + \mathrm{C}\,c' + \mathrm{D}\,d' \qquad \text{(in arc).} \quad [16.]$$

When only a few determinations are desired, the independent star numbers are usually more convenient; the formulas for this reduction are:

$$\alpha = \alpha_0 + f + \tau\,\mu + \tfrac{1}{15}\,g\ \sin.\,(\mathrm{G} + \alpha_0)\ \tan.\,\delta_0 + \tfrac{1}{15}\,h\ \sin.\,(\mathrm{H} + \alpha_0)\ \sec.\,\delta_0 \qquad \text{(in time).} \quad [17.]$$

$$\delta = \delta_0 + \tau\,\mu' + g\ \cos.\,(\mathrm{G} + \alpha_0) + h\ \cos.\,(\mathrm{H} + \alpha_0)\ \sin.\,\delta_0 + i\ \cos.\,\delta_0 \qquad \text{(in arc).} \quad [18.]$$

Tables giving τ, g, G, h, H, i follow those giving A, B, C, D, E in the "American Ephemeris." They are given for mean midnight of each day in the year for the meridian of Washington, D. C. (old observatory).

676. The instant of the beginning of the year is assumed by Bessel to be when the longitude or right ascension of the mean sun is $280° = 18^h\,40^m$, thus making $18^h\,40^m$ the sidereal time at which each year begins, and consequently simplifying all reductions. This is known as the *fictitious year.* The ephemerides are prepared

giving the values for τ based upon the above assumption, so that the computers using the ephemerides need not consider what the beginning of the year is. The dates used in most ephemerides are the astronomical, which begin at noon of the same civil date—i. e., are twelve hours later. The factor, τ, is the time of the astronomical date from the beginning of this fictitious year.

677. Parallax. The parallax of a star is the angle at the star between the lines from the points of observation. Ordinarily the term parallax is applied to that angle in the vertical plane formed by the lines from the center of the earth and the point of observation on the surface.

In the case of the fixed stars their distance is so great that the angle subtended by the radius of the earth is inappreciable, hence there is no correction for parallax. With the sun or other bodies of the solar system, however, it is different. As the sun is the only body upon which observations are made in this work, we have only to consider parallax in relation to it.

The sun's horizontal parallax for every tenth day during the year is given in the "American Ephemeris" (page 278), the horizontal parallax being that when the angle is the largest, or the sun in the horizon, and is merely the relation between the radius of the earth and the distance of the sun from the center of the earth.

If $x =$ horizontal parallax, $\rho =$ equatorial radius of the earth, D distance between the centers of the sun and earth, then

$$\sin. x = \frac{\rho}{D} \qquad [19.]$$

To obtain the parallax (p) for any altitude (h), we have, for all practical purposes,

$$p = x \cos. h = x \sin. z \qquad [20.]$$

The correction for parallax is always positive to h and negative to z, and less than the horizontal parallax.

678. Refraction is explained in connection with trigonometric leveling.

Various tables giving the corrections to observed altitudes or zenith distances have been computed, but the best are probably Bessel's.

If very accurate work is desired, the tables given in Chauvenet's "Astronomy," Vega's "Logarithms," or "Astronomical Observations of Washington Observatory" for 1845 may be consulted. They each have descriptions of the manner of using them.

If no tables are available, the refraction for small zenith distances may be obtained within a small fraction of a minute from

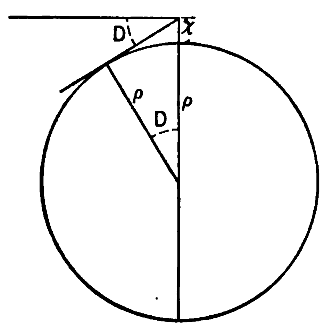
$$r = 57{\cdot}7'' \text{ tan. } z.$$

The most convenient tables for such work as described in this volume are the Tables I, XIV A and XIV B, given in Chauvenet's "Spherical and Practical Astronomy." These tables enable us to obtain the average refraction within 2″, which is probably about as close as we can expect to approach the true refraction effect anyway. Whenever greater accuracy is desired, the method of observation must be modified so as to eliminate the effect of refraction.

Large zenith distances should rarely be observed if good results are desired, as the refraction is very uncertain and variable near the horizon.

679. The Dip of the Horizon is the angle of depression of the sea horizon, due to the fact that the observer is above the curved surface of the earth. Whenever altitudes are measured from the sea horizon we must also have the elevation of the observer's eye above the sea level in order to correct for this angle of depression below the true horizon.

Fig. 466.



Let D = the dip (Fig. 466),

ρ = radius of the earth,

x = height of eye above the water in feet;

then $$\text{tan. D} = \frac{\sqrt{2\,\rho\,x + x^2}}{\rho}$$

As x^2 is very small compared with $2\,\rho\,x$, we may neglect it without appreciable error, and substitute

$$\text{D tan. } 1'' \text{ for tan. D}$$

hence $$\text{D} = \frac{1}{\text{tan. } 1''}\sqrt{\frac{2\,x}{\rho}}$$

Taking the average value for refraction, we obtain the final value for the dip:

$$D = 58{\cdot}82'' \sqrt{x \text{ in feet.}} \qquad [21.]$$

The correction to the observed altitude for dip is always negative.

RELATIONS OF DIFFERENT KINDS OF TIME USED IN ASTRONOMY.

680. The Sidereal and Solar Days. The *sidereal day* is the interval between two successive transits of the vernal equinox over the same meridian. The *sidereal time* at any instant is the hour angle of the vernal equinox reckoned from the meridian toward the west from 0^h to 24^h. The sidereal time at any place is 0^h when the vernal equinox is in the meridian of that place.

The *solar day* is the interval between two successive transits of the sun across any meridian, and the solar time at any instant is the hour angle of the sun at that instant. The solar day begins at any place when the sun is in the meridian of that place.

The *mean solar day* is the interval between two successive transits over the same meridian of a fictitious sun, called the mean sun, which is assumed to move uniformly in the equator at such a rate that it returns to the vernal equinox at the same instant with the actual sun.

Time reckoned with respect to the actual sun is called apparent time, while that reckoned with respect to the mean sun is called mean time. The difference between apparent and mean time, which amounts at most to about 16^m, is called the equation of time. This quantity is given for every day of the year in ephemerides or nautical almanacs.

The sidereal time when a star or other object crosses the meridian is called the *right ascension* of the object. The right ascension of the mean sun is also called the sidereal time of mean noon. This time is given for every day of the year in ephemerides for particular meridians, and can be found for any meridian by allowing for the difference in longitude.

The time to which ephemerides and most astronomical calculations are referred is the solar day, beginning at noon, and divided to

hours numbered continuously from 0^h to 24^h. This is called astronomical time, and such a day is called the astronomical day. It begins, therefore, twelve hours later than the civil day.

681. Relation of Apparent and Mean Time.

A = apparent time = hour angle of real sun;
M = mean time = hour angle of mean sun;
E = equation of time;

$$M = A + E.$$

In the use of this relation E may be most conveniently derived (by interpolation for the place of observation) from an ephemeris.

682. Relation of Sidereal and Mean Solar Intervals of Time.

I_m = interval of mean solar time;
I_s = corresponding interval in sidereal time;
r = the ratio of the tropical year expressed in sidereal days to the tropical year expressed in mean solar days

$$= \frac{366 \cdot 24222}{365 \cdot 24222} = 1 \cdot 0027379$$

$$I_s = r\,I_m = I_m + (r-1)\,I_m = I_m + 0 \cdot 0027379\,I_m, \qquad [22.]$$

or, if I_m is given in hours, $I_s = I_m + 9 \cdot 8565\,I_m$, where the correction $9 \cdot 8565\,I_m$ is in seconds of time,

$$I_m = r^{-1}\,I_s = I_s - (1 - r^{-1})\,I_s = I_s - 0 \cdot 0027304\,I_s. \qquad [23.]$$

Tables for making such calculations are usually given in ephemerides (see, for example, the "American Ephemeris," Tables II and III, near end of volume).

Frequent reference is made to the relations

24^h sidereal time = $23^h\,56^m\,04 \cdot 091^s$ solar time,
24 mean time = 24 03 56·555 sidereal time.

Interconversion of Sidereal and Mean Solar Time.

Let T_m = mean time at any place;
T_s = corresponding sidereal time, or R. A. of meridian of the place;
M = the R. A. of the mean sun for the place and date, or the sidereal time of mean noon for the place and date.
I_s, I_m, and r are same as above.

$T_s = M + I_s$ where $I_s = r\,I_m$. I_m and T_m being equal in this case, we have

$$T_s = M + r\,T_m = M + T_m + 0{\cdot}002738\,T_m. \qquad [24.]$$

To find the mean time, having given the sidereal time, we have

$$T_m = \frac{(T_s - M)}{r} = T_s - M - 0{\cdot}002730\,(T_s - M). \qquad [25.]$$

Example.—At Mount Ellen, Utah, August 19, 1891, longitude $2^h\,15^m\,01{\cdot}7^s$ W. of Washington, sidereal chronometer No. 2147 was compared with mean-time chronometer No. 2404 by noting the time of two consecutive coincidences as follows:

Chronometer No. 2147 reads	$17^h\,28^m\,34^s$	$17^h\,31^m\,35^s$
Chronometer No. 2404 reads	7 44 54	7 47 54·5

at same instants

If No. 2147 is fast on sidereal time $23{\cdot}03^s$, required the correction for No. 2404 on mean time.

Time of comparison by No. 2147, $\frac{1}{2}[(17^h\,28^m\,34^s) + (17^h\,31^m\,35^s)]$		= $17^h\,30^m\,04{\cdot}5^s$
Chronometer correction for No. 2147,		− 23·03
Sidereal time of comparison ($= T_s$),		17 29 41·47
Sidereal time of mean noon at Washington, August 19,	= $9^h\,51^m\,02{\cdot}43^s$	
Reduction to Mount Ellen − ($2^h\,15^m\,01{\cdot}7^s \times {\cdot}0027379$)	= + 22·19	
Sidereal time mean noon at Mount Ellen = M		= 9 51 24·62
Sidereal interval from mean noon,		= 7 38 16·85
Reduction to mean-time interval (Table II, "Amer. Ephemeris"),		− 1 15·08
Local mean time of the comparison,		7 37 01·77
Reading of chronometer No. 2404 at comparison,		= 7 46 24·25
Chronometer No. 2404 fast of local mean time, at $7^h\,46^m\,24^s$ by chronometer face, or at $7^h\,37^m\,02^s$ true mean time,		− 9 22·48

DETERMINATION OF THE TIME.

683. By **determination of the time** is always understood the determination of the error of the timepiece that the observer uses—either clock, chronometer, or watch.

In the following the word "clock" is used for any timepiece.

By means of astronomical observations upon celestial bodies whose positions are well known we may obtain the true local time, and having noted the instants of the observation by the clock we obtain its error referred to the meridian of the place where the observations were made.

If T represents the clock time,

T' the true time,

ΔT the correction to the clock,

we have
$$\Delta T = T' - T. \qquad [26.]$$

$$\text{For a chronometer} \left\{ \begin{matrix} \text{slow} \\ \text{fast} \end{matrix} \right\} \Delta T \text{ is} \left\{ \begin{matrix} + \\ - \end{matrix} \right\}.$$

If ΔT_0 and ΔT are the errors of a clock at the times T_0 and T, we can obtain the rate (R) of the clock per day, hour, or other unit, during the interval from T_0 to T, by

$$R = \frac{\Delta T - \Delta T_0}{T - T_0}, \text{ where} \qquad [27.]$$

$T - T_0$ is in same unit for which the rate is desired.

The correction ΔT to the clock at the time T may be found if the correction ΔT_0 at T_0 is known, and also the rate of the clock, from

$$\Delta T = \Delta T_0 + R(T - T_0) \qquad [28.]$$

These equations are based upon the assumption that the clock rate is uniform, which is rarely the case. If it is necessary to have as good a determination of the time as possible, we must make the observations in such a way as to have the correction for rate very small; also so close together that the rate may be considered uniform for the short interval.

There are a number of methods for determining the time, the most accurate being with the transit instrument adjusted in the meridian and used as described in Art. 684 and following.

When rougher determinations are sufficient we may employ any one of the following methods:

First Method.—If we have a transit adjusted in the meridian we may obtain the clock correction by noting the time of transit, T, of a star, since its hour angle at that instant is 0^h, and we may use its right ascension for the true time—i. e.,

$$T' = \alpha = \text{star's right ascension for the local meridian;}$$

hence we have

$$\Delta T = \alpha - T$$

where ΔT is the correction for the clock upon sidereal time.

The "American Ephemeris," "Berliner Jahrbuch," English and French nautical almanacs, contain the mean and apparent places for a number of the fundamental fixed stars, and hence it is always most convenient to select the stars from these catalogues and thus obtain the positions without having to compute them, besides being certain of getting stars whose places are well determined. When using one of these catalogues the observer's longitude must be known roughly with reference to the meridian for which the star places are given. For example, we have observed the sun at 22 hours, July 11, 1896, at a place, A, $3^h\ 20^m$ W. of the meridian of the ephemeris (Greenwich, we will suppose), and wish to find its position at the instant of observation. The "American Ephemeris" gives the position of the sun for apparent or mean noon for the meridian of Greenwich, hence we must find the Greenwich time which corresponds with the instant of the observation. This is equal to July 11th, $22^h + 3^h\ 20^m$ = July 12th, $1^h\ 20^m$, and we must interpolate between July 12th and 13th for $1^h\ 20^m$. If we wish the sun's position referred to mean time we use the right-hand page in the "American Ephemeris," and if we wish it in apparent time we use the left-hand page. These two pages are the first given for each month in the first part of the book.

If we have observed a fixed star on the meridian we have to interpolate for the difference in longitude between the place of observation and the meridian of the ephemeris, as the apparent places for the stars are given for the meridian transit at the meridian of the ephemeris.

Second Method. By Equal Altitudes of a Fixed Star.—The time of the meridian transit of a fixed star is the mean of the two times at which it is at the same altitude east and west of the meridian; hence, if no transit instrument is available, and we have an instrument that will enable us to obtain an altitude in any position, we may use the mean of the two times at which the star is at the same altitude just as though it were a meridian transit.

The sextant with an artificial horizon is usually preferred for this method of observation; but a theodolite, where the telescope may be clamped in an inclined position, may also be used. If only one altitude is observed we do not need to know what the altitude is, being careful to keep the instrument undisturbed between the eastern and western observations, so as to be certain that we actually have the same altitude in each case.

If accurate results are required we must get a number of observations on each side of the meridian. With a theodolite provided with a vertical circle it is best to note the time at several readings of the circle for the eastern observation, preferably at 10′ or 20′ divisions of the vernier, the former if the body is moving slowly and the latter if it is moving rapidly, and then repeat in reverse order when the star is on west side of meridian. By making part of the observations with telescope direct and part with it reversed we tend to eliminate some of the instrumental errors.

With the sextant and artificial horizon we measure twice the altitude of the star, since the image reflected from the surface of the artificial horizon is as much depressed below the horizon as the star is elevated above. If we could be certain of noting the exact time of transit (i. e., vertical transit) or coincidence, it would be more accurate to set the telescope, or mirrors of the sextant, in one position and note both transits without disturbing the instrument. But the error of observation is much greater than that of obtaining the same reading of the verniers, hence it is usually more accurate to note the transits for a number of positions by setting successively ahead of the star on even divisions, and then noting the time of transit for the western altitudes over the same positions in the reverse order, the last one of the eastern altitudes being the first one of the western. The mean of all is taken as the time of the meridian transit.

Example. March 15, 1856, equal altitudes of Spica were observed as below, the time being noted by a mean-time chronometer.

$$\text{Latitude} = -33^\circ\ 56'$$
$$\text{Longitude} = -\ 1^h\ 13^m\ 56^s \text{ from Greenwich.}$$

	Chronometer. East.	Sextant. Double Altitude.	Chronometer. West.
	$10^h\ 20^m\ 0{\cdot}5^s$	$104°\ 00'$	$2^h\ 40^m\ 38^s$
	20 28	10	40 10·5
	20 55	20	39 42
	$T_1 =$ 10 20 27·83		$T_2 =$ 2 40 10·17
$T = \frac{1}{2}(T_1 + T_2 =$	12 30 19·0		

From ephemeris,	$a = 13^h\ 17^m\ 37{\cdot}92^s$
Sidereal time mean noon from ephemeris,	$M =$ 23 32 53·22
Sidereal interval from mean noon,	$a - M =$ 13 44 44·70
Table II, ephemeris,	$-$ 2 15·12
Mean time,	$=$ 13 42 29·58
	$T =$ 12 30 19·00
Therefore	$\Delta T = +$ 1 12 10·58

Third Method. Equal Altitudes of Sun A. M. and P. M.— The declination of the sun changes so rapidly that we must obtain a correction to apply to the mean of two equal altitudes in order to obtain the time of the meridian transit or the instant of apparent noon. This correction is called *equation of equal altitudes.*

Let $\Delta\,\delta =$ one half the change in δ between the two observations, or the increase of δ from the meridian to the P. M. observation = decrease to the A. M. observation;

$T_o =$ mean of A. M. and P. M. observations $= \frac{1}{2}(T_1 + T_2)$;

$\Delta\,T_o =$ correction necessary to reduce T_o to clock time of apparent noon;

$t =$ one half the elapsed time between A. M. and P. M. observations;

Then $t + \Delta\,T_o$ and $t - \Delta\,T_o$ are the hour angles of the A. M. and P. M. observations respectively, the A. M. being reckoned toward the east; $\delta - \Delta\,\delta$ and $\delta + \Delta\,\delta$ are the sun's declinations at A. M. and P. M. observations respectively.

Substituting these values in the first of the fundamental equations [1], we obtain

$$\sin. h = \sin. \phi \sin. (\delta - \Delta\,\delta) + \cos. \phi \cos. (\delta - \Delta\,\delta) \cos. (t + \Delta\,T_o)$$
$$\sin. h = \sin. \phi \sin. (\delta + \Delta\,\delta) + \cos. \phi \cos. (\delta + \Delta\,\delta) \cos. (t - \Delta\,T_o)$$

By substituting the trigonometric equivalents for the quantities

in parentheses, subtracting the first equation from the second, and dividing by the coefficient of sin. ΔT_o, we have

$$\sin. \Delta T_o = - \frac{\tan. \Delta \delta \tan. \phi}{\sin. t} + \frac{\tan. \Delta \delta \tan. \delta}{\tan. t} \cos. \Delta T_o \qquad [29.]$$

This is the rigorous expression of the required correction, but as $\Delta \delta$ is always small, we may put $\Delta \delta$ for its tangent, ΔT_o for its sine, and unity for cosine ΔT_o, without any appreciable error. By dividing by 15 we get, then, ΔT_o in seconds of time, from

$$\Delta T_o = - \frac{\Delta \delta \tan. \phi}{15 \sin. t} + \frac{\Delta \delta \tan. \delta}{15 \tan. t}$$

If we use $\Delta' \delta$ as the hourly change for δ, which we obtain from the ephemeris for the instant of local noon of the date of observation, and use t in hours, then

$\Delta \delta = \Delta' \delta . t$, and we have the equation of noon.

$$\Delta T_o = - \frac{\Delta' \delta . t \tan. \phi}{15 \sin. t} + \frac{\Delta' \delta . t \tan. \delta}{15 \tan. t} \qquad [30.]$$

To facilitate the computation, tables have been prepared giving A and B, such that

$$A = - \frac{t}{15 \sin. t}, \text{ and } B = \frac{t}{15 \tan. t}$$

hence, if $a = A . \Delta' \delta \tan. \phi$ and $b = B . \Delta' \delta \tan. \delta$,

we have $\Delta T_o = a + b$ [31.]

When the sun is moving northward, $\Delta' \delta$ is positive when ϕ and δ are negative. A is positive when $t > 12^h$, and B positive when $t < 6^h$ or $> 18^h$. We obtain T from

$$T = \Delta T_o + T_o.$$

T is the clock time of the sun's meridian transit or apparent noon.

T′, the mean time of apparent noon, is equal to $0^h \pm E$, the equation of time obtained from the ephemeris for the instant of local noon.

Example. March 5, 1856, at the United States Naval Academy the sun was observed east and west of the meridian as follows:

A. M. East $= 1^h\ 8^m\ 26{\cdot}6^s$

P. M. West $= 8\ \ 45\ \ 41{\cdot}7$ Latitude $= 38° 59'$

$2t = (T_2 - T_1) = 7^h\ 37^m\ 15{\cdot}1^s$ Longitude from Washington $= - 2^m\ 16^s$

$T_o = \frac{1}{2}(T_1 + T_2) = 4^h 57^m\ 4{\cdot}15^s.$

	From ephemeris, $\delta =$	$-\ 5° 46'$	
	$\Delta' \delta =$	$+\cdot\ 58{\cdot}10''$	
Argument, $7^h 37^m$, Tables give	$A =$	$9{\cdot}4804_n$	$B = 9{\cdot}2151$
$\Delta' \delta = 58{\cdot}10''$	log. $=$	$1{\cdot}7642$	$1{\cdot}7642$
	log. tan. $\phi =$	$9{\cdot}9081$,	tan. $\delta = 9{\cdot}0047_n$
	log. a	$1{\cdot}1527_n$	log. $b = 9{\cdot}9840_n$
	$a =$	$-\ 14{\cdot}21^s$	$b = -\ 0{\cdot}96_s$
	$a + b = \Delta\, T_o =$	$-\ 15{\cdot}17^s$	
	$T_o =$	$4^h\ 57^m\ 04{\cdot}15^s$	
Clock time apparent noon $= T =$		$4^h\ 56^m\ 48{\cdot}98$	
Mean time of apparent noon	$= +$	$11\ \ 35{\cdot}11$	
Hence clock fast on mean time	.	$4^h\ 45^m\ 13{\cdot}87^s$	

Equal Altitude of Sun on Afternoon of One Day and Morning of Next. By substituting $+$ for $-$ for first term of equation [30] we will obtain the correction for midnight instead of for noon, hence we may use equation [30] by merely changing the sign of A.

Correction for Small Inequalities in Altitudes. If the refraction is different at the two observations, equal apparent altitudes will not be equal true altitudes; or if part of the observations are lost in the second observation, we will have two unequal altitudes.

If Δh is the difference obtained by subtracting the first true altitude from the second true altitude, we can obtain the correction, $\Delta' T_o$, to T_o from

$$\Delta' T_o = \frac{\Delta\, h \cos. h}{30 \cos. \phi \cos. \delta \sin. t} \qquad [32.]$$

The advantages of the methods of equal altitudes are that no corrections are required for parallax, refraction, semidiameter, or instrumental errors, nor is a knowledge of the latitude or declination required, except roughly for observations on the sun.

Fourth Method. By a Single Altitude. An altitude (or zenith distance) of a celestial body may be observed and the time noted by the clock, or, for greater precision, observe several altitudes in quick succession, assuming that the mean of the altitudes corresponds with the mean of the times noted. This is not strictly true, since

the body moves on a curved line. If the observations are not very near the meridian nor extend over an interval of more than a very few minutes, the error in the above assumption is not appreciable as compared with errors of observation or uncertainty in the refraction.

If ζ is the observed zenith distance corrected for known instrumental errors (such as index correction for the sextant, etc.), and

z is true zenith distance,

r the refraction,

p the parallax,

s the sun's semidiameter, to be used when only one of the sun's limbs is observed, we have

$z = \zeta + r - p \pm s$ for the sun, and

$z = \zeta + r$ for a star.

Altitude, $h = 90° - z$.

$$s \text{ is} \left\{ \begin{matrix} + \\ - \end{matrix} \right\} \text{when the} \left\{ \begin{matrix} \text{upper} \\ \text{lower} \end{matrix} \right\} \text{limb is observed.}$$

From equations [6] and [7] we find

$$\cos. t = \frac{\cos. z - \sin. \phi \sin. \delta}{\cos. \phi \cos. \delta} \qquad [33.]$$

$$\tan.^2 \tfrac{1}{2} t = \frac{\sin. (s - \phi) \cos. (s - \delta)}{\cos. s \cos. (s - z)} \qquad [34.]$$

when $$s = \tfrac{1}{2} (\phi + \delta + z) \qquad [35.]$$

If the greatest accuracy is desired the second formula should be used.

For the best results the celestial body should be observed on the prime vertical—i. e., east or west of the observer, provided it is not so low as to introduce abnormal refractions.

Observations within 10° or 15° of the horizon are not very reliable, and should never be taken if they can be avoided.

The least favorable position of the star is when it is near the meridian, as the change in altitude is so small that it is difficult to get accurate measures or note the time very closely.

684. The Transit Instrument. The accompanying cuts represent several of the small portable transits used by the United States

Coast and Geodetic Survey. Various other forms have been used from time to time, but have gradually been replaced by the form shown in Fig. 468, when both latitude and time are required. When a very light instrument is required, one similar to that shown in

Fig. 467.

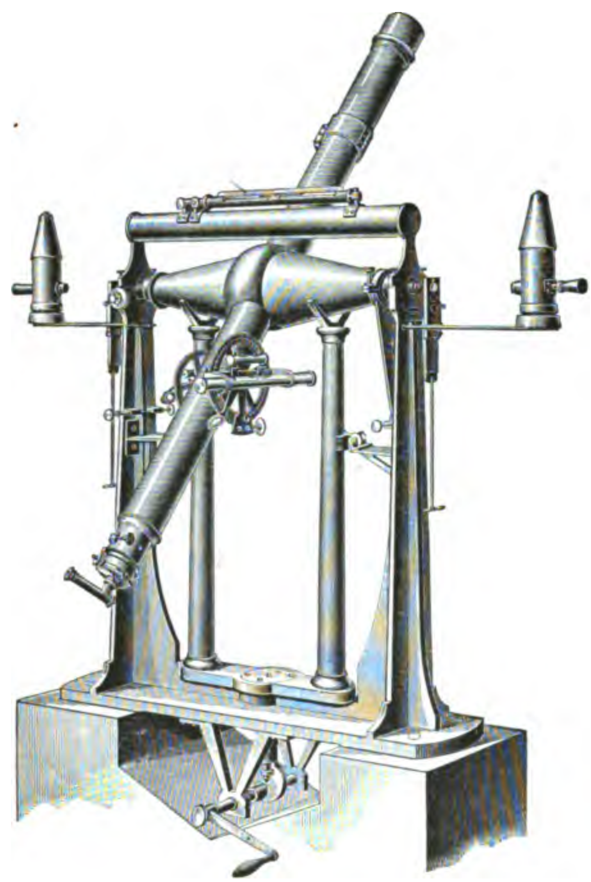
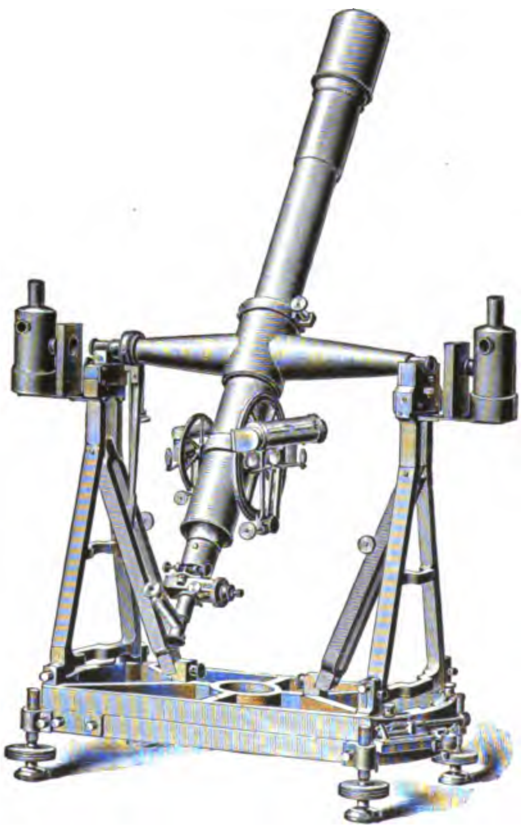


Fig. 467 is used, but of lighter construction. Fig. 467 is a representation of the instrument formerly used for the telegraphic determination of longitude, a work requiring the greatest accuracy The instrument now used for this class of work is similar in construction, but lighter. It has a telescope of 95 centimetres (37½

inches) focal length, 82 millimetres ($3\frac{1}{4}$ inches) objective, with a magnifying power of about 100 diameters. The telescope is reversed by means of false wyes which are raised by a crank and cam arrangement at the base, thus lifting the telescope from the wyes. The apparatus is then twisted 180° in azimuth about an axis at the

FIG. 468.



base and lowered, thus reversing the telescope without jarring the instrument. This instrument is used merely for time observations, having no micrometer attachment.

685. The Meridian Telescope. Fig. 468 represents the *meridian telescope*, or universal instrument. It is used for the determination

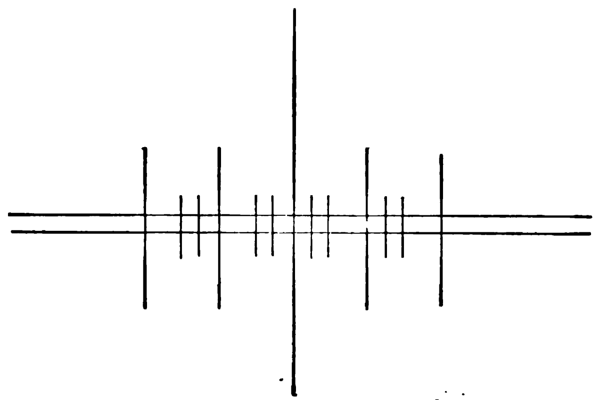
of time, latitude, and azimuth; the uprights are pivoted at the base and fold down when packed for shipment, thus making a very convenient and portable instrument. The base is split in the middle, horizontally, and the upper part revolves around a short vertical axis in the center of the base, thus giving the instrument all the motions of a zenith telescope. Whenever time observations are to be made, the upper part of the base is clamped to the lower by means of screw bolts at the sides, and the instrument becomes a fairly rigid and stable transit. These instruments are usually smaller than the simple transit. Two sizes are used upon the Coast and Geodetic Survey, the proportions being relative to the size of the telescopes. The smaller instruments have telescopes of 66 centimetres (26 inches) focal length, 57 millimetres ($2\frac{1}{4}$ inches) objective, with magnifying powers 50 to 70 diameters, and the larger 79 centimetres (31 inches) focal length, 63 millimetres ($2\frac{1}{2}$ inches) objective, and magnifying powers 60 to 90 diameters. All these meridian telescopes are fitted with micrometer eyepieces which are capable of being revolved about the axis of the telescope, so they can be used for either vertical or horizontal measurements. The micrometer heads are divided into 100 equal divisions. A toothed rack on the reticule inside the telescope is used to mark whole turns of the screw.

686. Diaphragms. In order to increase the accuracy of marking the transit of a star a number of vertical lines are used in the focus of the telescope instead of a single line. They are cut on a very thin glass film, and are usually arranged as shown in Fig. 469, 13 lines in tallies of 1, 3, and 5. When observations are made with the chronograph for recording the transits, all the lines are used excepting the two outside lines—i. e., the 11 in the middle. When the eye-and-ear method is used only the long lines are observed upon. The small intervals are usually from $2''$ to $3\frac{1}{2}''$ for equatorial stars. Formerly 25 lines were used in chronographic observations, but the inaccuracies due to the fatigue of so many observations more than offset the gain in accuracy obtained by increasing the number of transits beyond about 9 or 11. Two horizontal lines are usually placed

in the center of the diaphragm to enable the observer to use the same part of the vertical lines for each star, thus eliminating the effect of any irregularity in the lines or error of their verticality.

To avoid introduction of parallax in case the eye should not be directly over the line when transited by the star, the field of view is

Fig. 469.



limited by a slide with a circular opening, which is moved by the observer successively over the lines as traversed by the star. The telescopes are provided with prismatic eyepieces, thus permitting the observation of zenith stars.

Small finders, reading by verniers to minutes, are attached to the sides of the telescope. They are usually graduated so as to read zenith distances and permit quick settings for stars.

The striding level vial is filled with ether, hermetically closed, and supplied with a chamber in one end to admit the regulation of the length of the bubble for changing temperatures. The sensitiveness of the level is usually such that a change of one second of arc corresponds to a motion of the bubble of about a millimetre. The tube containing the level vial is provided with two motions (one vertical and the other horizontal), to admit its being made parallel with the line joining the points of support of the level.

The adjustment of the striding level and also that for the col-

limation of the telescope of the transit instrument are made in the same way as for the theodolite, Art. 603.

687. To make the Lines Vertical. In the transit the observing lines are usually placed in a vertical position by the maker and can not be disturbed, but in the meridian telescope, where the micrometer box is movable around the axis of the telescope, the observer must place the lines in a vertical position. This may be done in two ways: First, by making them parallel to a plumb line or vertical pole (the axis having been previously leveled), or, second, bisecting a distant point and noting whether the line continues to bisect this point when the telescope is slightly elevated or depressed. The adjustment is made by revolving the micrometer box, which is provided with a clamp to hold it in position, and a slow-motion screw to perfect the adjustment.

688. To adjust the Finders. Point the telescope upon some easily recognized point, and bring it to the center of the field or where it is desirable to have the stars transit; bring bubble of finder level to the center, and note the reading of the vernier. Then reverse the telescope and repeat the pointing and reading. If the two readings of the vernier agree the finder is in adjustment. If they do not agree, change the vernier so that it reads halfway between the two readings and then bring the bubble of the level into the center by means of its adjusting screws, being careful to keep the telescope pointed upon the object at the same time. If the circle is not graduated to read zenith distances or altitudes for each position of the telescope, one of the readings will be the complement of the other instead of being equal to it, and must be so considered in making the correction.

689. To Adjust the Instrument in the Plane of the Meridian. The sidereal chronometer or clock is generally used with the astronomical transit, and the observations are to determine its error. If a mean-time chronometer is used, we must consider it a sidereal chronometer with a very large rate.

When we occupy a new station we obtain its geographic position

as accurately as possible from maps, or other sources of information, and start the chronometer as near local sidereal time as can be obtained without making special observations to determine it. If the station is near a telegraph office it is always possible to get standard mean time very accurately, and this may be converted to local sidereal time with an accuracy depending principally upon our knowledge of the longitude of the station.

If we have no means of determining the time we may guess at it, and gradually obtain the true local time by observation with the transit while getting it into the meridian.

If we have a very rough determination of the time, it is best to set the transit upon α Ursæ Minoris (Polaris), allowing a little for its position with reference to the pole as shown by the right ascension obtained from the ephemeris (considering the chronometer correct), knowing that its extreme range either side of the pole is only about $1\frac{1}{4}°$. Polaris can always be seen in the northern hemisphere, and is easily found in the field of the telescope. It may usually be seen at any hour during the day with telescopes magnifying 75 or more diameters, and nearly all day with a 50-diameter magnifying power, provided the atmosphere is fairly clear and steady.

Having now set roughly in the meridian, we level the axis and note the transit of a zenith star. Obtaining the right ascension of this star from the ephemeris, we get a very fair error of the chronometer by taking the difference between the chronometer time of transit and the right ascension of the star. With this better chronometer error we compute the chronometer time at which a close circumpolar star will cross the meridian, and then follow this star with the telescope until the chronometer reaches this computed time. The telescope should then be very nearly in the meridian, the accuracy depending upon how near the zenith the time star was and the rapidity of the circumpolar star. If now another zenith star is taken (always being careful to have the telescope axis level), a new correction will be obtained for the chronometer, which is nearer the true correction than the other. If the time stars are very near the zenith (within 3° or 4°, say), and the circumpolar or azimuth star within 10° of the pole, we may conclude that the error of the chronometer is known probably within a second. Taking

now another azimuth star as near the pole as possible (and preferably below it), we can set the instrument very accurately in the meridian by using the slow-motion screw, and are ready, after making the axis of the telescope horizontal, to begin the observations proper for time.

690. Method of Observation and Selection of Stars. It is practically impossible to adjust a transit so as to entirely remove the errors of collimation and azimuth, hence the observations are made in such a way as to determine these errors.

A time set, therefore, consists of *time stars* and *azimuth stars*, observed in direct and reversed positions of the telescope to obtain the error of collimation.

Time stars are those near the zenith or toward the equator, and move rapidly so that they appear to cross the lines in the telescope in an instant.

Azimuth stars are those with large zenith distances, and usually near the pole, either above or below. Those within about 8° of the pole, however, are not taken, excepting in an emergency, as they move so slowly it takes too long for them to cross the field of the telescope.

A complete time set consists of at least four stars, two being time stars and two azimuth stars. Ordinarily, however, eight or ten stars are observed, two being azimuth stars; one azimuth star and half of the time stars with telescope direct, and the others with telescope reverse.

In order to eliminate the effect of any error in the determination of the azimuthal deviation it is desirable to so choose the time stars that the sum of the A's (see azimuth correction, page 172) for telescope direct balances the sum of A's for telescope reverse, and that the time stars in each position be on each side of the zenith, and such that

$$\tfrac{1}{2}(\tan.\delta + \tan.\delta') = \tan.\phi, \qquad [36.]$$

δ being the mean of the declinations of the time stars south of the zenith, δ' the mean of those north, and ϕ the latitude of the station.

Owing to the small number of stars available for time observa-

tions it will rarely be possible to fulfill these conditions without extending the time too much. The observations should be made as rapidly as possible, so the condition of the instrument will not be likely to change materially during the work.

The striding level is placed on the pivots and read for each star (or as often as possible), to determine the inclination of the horizontal axis. *A level reading* always consists of a reading of both ends of the bubble with the level in one position, and also immediately after with it reversed end for end, so as to eliminate the error of adjustment in the level.

The following list shows an average time set and a convenient arrangement for use in the observatory—$\phi = 37° 47'$:

STAR.	MAG.	R. A.	δ	λ	
		h m s	° ′	° ′	
γ' Androm................	2·4	Level. 57 31	41 50	85 57	N.
β Trianguli................	3·0	Level. 2 03 21	84 30	86 43	S.
γ Trianguli................	4·3	Level. 11 08	83 22	85 35	S.
ξ^2 Ceti....................	4·5	Level Set. 2. 22 38	8 00	60 13	S.
36 Cassiop................	5·6	Level N. 28 09	72 22	55 25	N.
ν Arietis..................	5·6	32 55	21 31	73 44	S.
μ Ceti.....................	4·0	Level S. 39 19	9 40	61 53	S.
41 Arietis.................	3·8	43 52	26 50	79 03	S.
47 Cephei.................	6·0	Level S. 52 16	79 01	48 46	N.
ρ Persei..................	4·0±	Level N. 58 31 Level N.	88 26	89 21	N.

There are two methods of marking the transits, the *eye-and-ear method* and the *chronographic method.*

In the eye-and-ear method the observer picks up the beat of the chronometer and counts the beats mentally until the star crosses the line, when he notes the instant, estimating tenths of seconds. As most chronometers beat half seconds, the observer merely estimates fifths of the intervals between beats. An experienced observer will rarely be in error 0·2 of a second in estimating the time over a single line for stars near the equator.

In the chronographic method the observer registers the instant when the star crosses the line by pressing a key which closes or breaks an electric circuit, and thus makes a record on the revolving cylinder of a chronograph. A chronometer is recording regularly upon this cylinder, hence the time of transit of a star over the line is obtained by measuring the position of the break on the sheet between the two chronometer breaks next adjoining it. A glass scale with diverging lines is used to read the fractions of seconds, the outside lines of the scale being made to coincide with the breaks made by the chronometer.

691. Record. The following sample of a record of the United States Coast and Geodetic Survey work is a convenient form for transit observations. When the chronograph is used the original record is upon the chronograph sheet, and a separate sheet for the level record, but when the sheet is read the record is kept the same as in this illustration.

Station, Lafayette Park. Date, November 25, 1896. Instrument, Transit No. 4. Observer, O. B. F. Recorder, O. B. F. Chronometer, Hutton No. 211.

γ' ANDROM.		β TRIANGULI.		γ TRIANGULI.		ξ^2 CETI.		36 CASSIOP.		ν ARIETIS.	
W.		W.		W.		W.		W.		E.	
W. E.		W. E.		W. E.		W. E.		W. E.		W. E.	
N. 87·6 85·0		S. 87·2 85·6		87·2 86·0		87·8 85·8		87 8 86·2		87 0 86·8	
87·0 85 7		87·5 85·5		87·1 86·1		87·7 86·0		88·6 85·8		88·8 85·8	
+3·9		+3·6		+2 2		+3·2		+4·9		+2·7	
				Mean = +3·56d							
B=+1·84		B=+1·21		B=1·19		B=0·88		B=+2·72		B=+1·08	
		b=+0·06						Pivot inequality inappreciable.			
(1 div. of level=1·01″=0·0674s ¼(0·0674s)=0·0169s)											
1 56 46·0		2 02 38·2		2 10 25·2		2 21 57·8		2 26 57·8		2 32 13·2	
49·6		41·5		28·2		22 00·8		27 05·8		15·9	
53·0		44·5		31·4		02·9		14·6		18·7	
59·9		50·9		37·5		08·0		31·8		24·2	
57 03 4		53·9		40·6		10·7		39·7		26·9	
06 8	06·8	56 9	56·9	43·7	43·7	13·3	13·3	48 3	48·3	29·7	29·7
10 2	13·6	03 00·0	113·9	46·8	87·4	15·8	26·5	56 6	96·8	32·5	59·4
13·6	13·5	03·2	114·1	49·7	87·2	18·4	26·4	28 05·1	96·4	35 2	59·4
20·6	13·6	09·6	114·1	55·9	87 8	23·6	26·5	22·0	96·6	40·8	59·5
24 0	13·6	12·6	114 1	59·0	87·2	26·3	26·6	30·5	96·8	43·5	59 4
27·4	13·4	15 7	113·9	11 02·1	87 8	28·9	26·7	39·3	96·6	46·2	59·4
1 57 06·77		2 02 57·00		2 10 43·65		2 22 13·27		2 27 48·23		2 32 29·71	
K = −·02		−·02		−·02		−·02		−·06		−·02	
δB.=+·08		+·07		+·07		+·05		+·16		+·06	
R = ·00		00		·00		·00		·00		·00	
1 57 06·83		2 02 57·05		2 10 43·70		2 22 13·30		2 27 48·33		2 32 29·75	
1 57 35·94		2 03 26·24		2 11 12·88		2 22 42·38		2 28 17·64		2 32 59·45	
+29·11		+29 19		+29·18		+29·08		+29·31		+29·70	

μ CETI.		41 ARIETIS.		47 CEPHEI.		ρ PERSEI.	
E.		E.		E.		E.	
W. E.							
38·0 36·3				38·7 35·3		37·6 36·9	
37·2 37 1				38·3 36·2		38·7 35·6	
+1·8				+5·5		+3·8	
		Mean = +3·45d					
		b = +0·058					
B = +0·90		B = +1·10		B = +3·94		B = +1·23	
2 38 38·6		2 43 09·6		2 50 35·3		Not observed.	
41·3		12·5		49 2			
43·8		15·3		51 02·6			
49·1		21·1		29·8			
51 5		24·1		42·5			
54·2	54·2	26·8	26·8	57·2	57·2		
56·9	108·4	29·6	53·7	52 10·2	52·7		
59 5	108·6	32·5	53·6	23·3	53·1		
39 04·6	108·4	38·5	53·8	50·3	52·9	2 58 19·4	
07·3	108·6	41·4	53·9	03·7	52·9	22 7	
09·8	108·4	44·2	53·8	17·5	52·8	25 9	
2 38 54·24		2 43 26·88		2 51 56·51		2 58 06·27	
−·02		−·02		−·09		−·02	
+·05		+·06		+·23		+·07	
·00		·00		·00		·00	
2 38 54·27		2 43 26·92		2 51 56·65		2 58 06·32	
2 39 23·81		2 43 56·70		2 52 29·06		2 58 36·19	
+29·54		+29·78		+32·41		+29·87	

Reduction of broken transit of ρ Persei *to center.*	Equatorial intervals.
δ=38° 26′	−15·418
	−12·847
Cos. δ= 9·89395	−10·232
Log. (38·580)=1·58580	
	−5·166
1·69185	−2·570
Log. 3= 0·47712	+0·011
	+2·553
1·21473	+5·134
Red'n.=−16·40s	+10·270
	+12·850
	+15·410

Mean of observed lines $=2^h\ 58^m\ 22{\cdot}67^s$
Mean of equatorial intervals of observed lines= +38·580

692. Pivot Inequality. The pivots at the ends of the horizontal axis of the telescope are rarely turned to the same diameter, hence when the striding level is set upon them it does not record the actual inclination of the center of the axis. Observations are made to determine the amount of this inequality, and a correction for it applied to each level reading. If the same pivot gives level readings too great (is high) both before and after reversal of the telescope, half the difference between the two level corrections is the effect due to the difference of diameter of the pivots; if the east pivot shows high $\left\{\begin{matrix}\text{before}\\\text{after}\end{matrix}\right\}$ reversal and the west pivot high $\left\{\begin{matrix}\text{after}\\\text{before}\end{matrix}\right\}$ half the sum of the level corrections is the effect. Half the effect is the correction to the level corrections for inequality of pivots (since the transit's axis passes through center of pivots) and is $\left\{\begin{matrix}-\\+\end{matrix}\right\}$ to $\left\{\begin{matrix}\text{large}\\\text{small}\end{matrix}\right\}$ pivot.

Example of Record and Computation of Inequality of Pivots.— Let b_w and b_e designate the inclination, as given by the level read-

ings, for clamp west and east respectively; β_w and β_e the same, when corrected for pivot inequality p; then

$$p = \frac{\beta_e - \beta_w}{4} \text{ and } \begin{cases} \beta_w = b_w + p \text{ for clamp west,} \\ \beta_e = b_e - p \text{ for clamp east,} \end{cases}$$

supposing the V-bearings of instrument and level to have the same angular opening, and the pivot to be circular in form.

OBSERVATIONS FOR INEQUALITY OF PIVOTS OF TRANSIT NO. 4.

Station, Seaton, Washington. G. W. D., observer. June 19, 1865.

Altitude.	Time.	Temperature, Fahr.	Clamp West. Object glass S. Level. W. end.	E. end.	$\frac{1}{2}(\Sigma w - \Sigma e)$ b_w	Clamp East. Object glass N. Level. W. end.	E. end.	$\frac{1}{2}(\Sigma w - \Sigma e)$ b_e	$\frac{b_e - b_w}{4} = p$
°	h. m.	°	d.	d.	d.	d.	d.	d.	d.
55	10.30 A. M.	73	60·0	64·0	+0·600	59·0	65·2	−0·425	−0·256
			65·2	58·8		64·0	59·5		
50	45 "	72	65·0	59·0	+0·950	64·0	59·5	−0·250	−0·300
			60·8	63·0		59·0	64·5		
45	50 "	72·5	60·8	63·0	+1·450	59·5	64·0	−0·125	−0·394
			66·0	58·0		64·0	60·0		
40	11.00 "	72·8	65·0	58·8	+1·050	64·0	60·0	−0·175	−0·306
			61·0	63·0		59·3	64·0		
35	05 "	73	60·5	63·0	+1·200	59·2	64·0	−0·575	−0·444
			65·5	58·2		63·0	60·5		
30	15 "	73·2	63·5	58·0	+1·000	63·0	60·5	−0·125	−0·281
			60·5	62·0		60·0	63·0		
25	20 "	73·8	60·5	62·0	+1·125	59·0	63·0	−0·125	−0·312
			64·0	58·0		62·5	59·0		
20	30 "	74	62·0	57·5	+1·500	62·0	59·0	0·000	−0·375
			60·8	59·3		59·0	62·0		
15	40 "	74·5	60·0	60·0	+1·375	58·0	61·0	−0·125	−0·375
			62·5	57·0		60·5	58·0		
10	50 "	75	62·0	57·0	+1·750	60·5	58·0	0·000	−0·437
			60·0	58·0		58·0	60·5		
5	12.00 M.	76	58·5	59·0	+1·250	58·0	60·5	−0·125	−0·344
			61·5	56·0		59·0	57·0		
0	0.10 P. M.	76	60·0	57·0	+0·875	59·0	57·0	0·000	−0·219
			58·5	58·0		57·0	59·0		

Value of one division of level=1·05″.

$$\text{Mean value of } p = -\,0{\cdot}337^d \pm 0{\cdot}013^d$$
$$= -\,0{\cdot}024^s \pm 0{\cdot}001^s$$

693. Value of Level Division. *Example of the Determination of the Value of One Division of the Level.*

Station, Harris, August 23, 1855.—Observations for value of one division of level B of zenith telescope No. 2. Collimator eight feet from object glass. Value of one division of micrometer screw (mean of 4 sets)=0·448″. Observer, G. W. D.

NOTE.—Only a part of the observations made are here given.

NO.	TEMPERATURE.	MICR. TURNS.	LEVEL READING.		DIFFERENCE OF READING.		VALUE OF 1 DIV. OF LEVEL.	Δ	Δ²
			NORTH.	SOUTH.	IN MICR.	IN LEVEL.			
	° *Fahr.*		*d.*	*d.*	*d.*	*d.*	*d.*		
1	66·2	18·94	34·2	1·8	64	25·2	2·54	0·01	·000
		18·30	9·0	27·0					
2	....	18·26	34·7	1·4	61	24·55	2·48	0·07	·005
		17·65	10·0	25·8					
3	....	17·64	36·0	0·0	66	27·1	2·44	0·11	·012
		16·98	8·8	27·0					
4	....	16·95	35·5	0·0	73	30·0	2·43	0·12	·014
		16·22	5·5	30·0					
5	66·5	16·22	34·0	1·5	74	28·45	2·60	0·05	·002
		15·48	5·6	30·0					
6	....	15·43	34·3	1·2	76	30·85	2·46	0·09	·008
		14·67	3·5	32·1					
7	....	14·62	31·0	4·8	81	32·35	2·50	0·05	·003
		13·81	− 1·5	37·0					
8	....	13·77	33·4	2·2	67	25·0	2·68	0·13	·017
		13·10	8·2	27·0					
9	....	13·07	35·0	0·2	71	27·55	2·58	0·03	·001
		12·36	7·5	27·8					
10	67·0	12·33	35·0	0·6	67	25·3	2·65	0·10	·010
		11·66	9·5	25·7					
11	....	11·65	30·5	4·8	60	22·35	2·68	0·13	·017
		11·05	8·0	27·0					
12	....	11·00	33·0	1·9	68	26·25	2·59	0·04	·002
		10·32	6·8	28·2					
Mean......							2·55	Sum	0·091

One division of level B=2·55×0·448″=1·14″ at temperature 66 6° Fahr., with a probable error of

$$\sqrt{\frac{0{\cdot}455\times0{\cdot}091}{12\times11}}=\pm0{\cdot}018d=\pm0{\cdot}01''$$

694. Determination of Equatorial Intervals of Lines. When all the lines are observed, the mean is taken and considered the transit of the star; but when a line is lost, we must know its position with reference to the mean of all, in order to correct the mean of the observed lines to what it would have been if no lines had been lost. For this purpose observations are made to determine the equatorial intervals of the lines—i. e., the interval between each line and the mean of all, as would be shown by an equatorial star (with $\delta = 0°$)

crossing the field. For accuracy, stars of great declination are observed, as they move slower and are marked more accurately. The equatorial intervals are then determined as follows: Let

$t_1, t_2, t_3, \ldots . t_n$ be the observed times of transit over the successive threads;

t their mean,

$i_1, i_2, i_3, \ldots . i_n$ their equatorial intervals from the mean thread; and

δ the declination of the star:

$$t = \frac{1}{n}(t_1 + t_2 + t_3 \ldots . + t_n)$$

$$i_1 = (t_1 - t) \cos. \delta$$

$$i_2 = (t_2 - t) \cos. \delta$$

$$\text{etc.}$$

$$i_n = (t_n - t) \cos. \delta$$

also

$$0 = i_1 + i_2 + i_3 \ldots . + i_n$$

The intervals of the threads $\left\{ \begin{matrix} \text{east} \\ \text{west} \end{matrix} \right\}$ of the mean thread will then be $\left\{ \begin{matrix} - \\ + \end{matrix} \right\}$ at upper culmination.

For stars within 10° of the pole (as for δ Urs. Min., 51 Cephei, Polaris, and λ Urs. Min.), use the formulas:

$$i_1 = (t_1 - t) \cos. \delta \sqrt[3]{\cos. \tau_1}$$

$$\text{etc.}$$

$$i_n = (t_n - t) \cos. \delta \sqrt[3]{\cos. \tau_n} \qquad [37.]$$

where $\tau_1, \tau_2, \tau_3, \ldots . \tau_n$ the hour angles of the circumpolar star for the successive threads.

When the chronometer has a large rate the intervals require to be corrected for it.

Incomplete Transits.

When the star was not observed on some of the threads, the time of transit over the mean of all the threads may, by means of the known intervals of the threads, be found as follows:

t = mean of observed times

$$+ \frac{\text{sum of equatorial intervals of missed threads} \times \text{sec.}\ \delta}{\text{number of observed threads}} \qquad [38.]$$

If the transit over one, or a few threads only, is observed, we may use the formula

$$t = \text{mean of observed times} - \frac{\text{sum of equatorial intervals of observed threads} \times \sec.\,\delta}{\text{number of observed threads}}.$$

In reducing broken transits of a circumpolar star, use $i_1 \sqrt[3]{\sec.\,\tau_1}$, $i_2 \sqrt[3]{\sec.\,\tau_2}, \ldots\, i_n \sqrt[3]{\sec.\,\tau_n}$ in the place of the equatorial intervals $i_1, i_2, \ldots\, i_n$.

Apply also a correction for rate, if necessary.

695. Computation of Chronometer Correction. Having obtained the inequality of the pivots, the value of one division of the striding level, and, from observation, the times of transit, t, of a set of stars selected, as explained on page 163, the instrument having been previously adjusted as closely in the meridian as possible, we must compute the instrumental corrections necessary to eliminate the effect on each t of (1) pivot inequality and inclination of the horizontal axis, (2) diurnal aberration, (3) chronometer rate, and, finally, the corrections for collimation and azimuth.

1. *Inclination.*—Let w and e, w' and e' be west and east end readings of level bubble before and after reversal respectively, d the value of one division of level in seconds of arc, b the level correction representing the inclination of the axis, $\left\{\begin{matrix}+\\-\end{matrix}\right\}$ when $\left\{\begin{matrix}\text{west}\\\text{east}\end{matrix}\right\}$ end of axis is too high; then

$$b = \tfrac{1}{4}\left\{(w+w') - (e+e')\right\}\frac{d}{15} = \left\{(w+w') - (e+e')\right\}\frac{d}{60}$$

in seconds of time. In order to get the effect of b upon each star we must multiply it by $\frac{\cos.\,z}{\cos.\,\delta}$ where z and δ are the zenith distance and declination of the star respectively. z is always positive. δ is taken negative south of the equator, and positive north. Hence $\frac{\cos.\,z}{\cos.\,\delta} = \cos.\,z \sec.\,\delta$ is positive excepting when δ is greater than 90°, or the star is at lower culmination.

Cos. z sec. $\delta = \mathrm{B}$ may be tabulated for the arguments z and δ, and the factor B obtained for each star.

The correction for inequality of pivots (p) is best applied directly to the level correction (b) before it is multiplied by B, since both must be multiplied by B, or, in other words, make the level difference what it would have been had the pivots been equal. If we wish to apply the pivot inequality to $(w + w') - (e + e')$ we must multiply p by 8, as we are using four times the actual movement of the bubble, and the effect upon the level is double the value of p. p is the inclination of the center of the axis with reference to either line of supports, and therefore $2p$ is the inclination of upper surface of the pivots referred to the lower.

2. *Diurnal Aberration.*—When great precision is desired we must apply a small correction to each star for diurnal aberration, K. This is always a very small correction, and is negative to all stars at upper culmination and plus at lower culmination. In amount it is

$$K = 0{\cdot}0206 \cos. \phi \sec. \delta \qquad [39.]$$

It may be tabulated for the arguments ϕ and δ, and the correction for each star applied at the same time as B b, the correction for inclination of axis.

3. *Chronometer Rate.*—It is evident that the chronometer correction obtained from the first star of a time set will not be the same as that from the last where the chronometer has a rate, hence we must apply a small correction to each of the t's to reduce them to what they would have been if the chronometer had no rate. Let T_1 be an assumed time near the middle of the time set, R the chronometer rate, and t the time of transit of the star, then the correction to each t is

$$\pm (t - T_1) R; \left\{ \begin{matrix} + \\ - \end{matrix} \right\} \text{when chronometer is} \left\{ \begin{matrix} \text{losing} \\ \text{gaining} \end{matrix} \right\}.$$

Thus far we have been able to determine the exact corrections to be applied to the t's, but in order to eliminate the effect of the errors of collimation and azimuthal deviation we have first to obtain them from the observations themselves.

696. Collimation Correction. If c is the amount of the error in collimation in seconds of time, then the correction to t is

$$\pm c \sec. \delta = \pm c \, C \text{ when } \sec. \delta = C.$$

At upper culmination c is $\left\{\begin{matrix}+\\-\end{matrix}\right\}$ when the mean of the threads is $\left\{\begin{matrix}\text{east}\\\text{west}\end{matrix}\right\}$ of the line of collimation for any assumed position, say for clamp or lamp west. At lower culmination the sign is reversed. By reversing the telescope the sign of c is reversed. This sign, however, is usually applied to C, being generally taken $\left\{\begin{matrix}+\\-\end{matrix}\right\}$ for clamp $\left\{\begin{matrix}\text{east}\\\text{west}\end{matrix}\right\}$ for stars at upper culmination, the reverse sign being taken when star is at lower culmination.

697. Azimuth Correction. Let a = azimuth error in seconds of time, then the correction to t is

$$\pm\, a \sin. z \sec. \delta = a\,\text{A} \text{ where } \sin. z \sec. \delta = \text{A}.^*$$

A is always positive except for stars between the zenith and pole.

a is $\left\{\begin{matrix}+\\-\end{matrix}\right\}$ when the line of collimation is $\left\{\begin{matrix}\text{east}\\\text{west}\end{matrix}\right\}$ of south.

698. Chronometer Correction. t is the mean of the observed transits of a star; let T be the result after applying the first three corrections above—i. e., $(b \pm p)$ B, R, and K, whence

$$\text{T} = t + (b \pm p)\,\text{B} + \text{R} + \text{K}. \qquad [40.]$$

The chronometer correction uncorrected for collimation and azimuth is then

$$\alpha - \text{T},$$

and the final chronometer correction, Δ T, is

$$\Delta\,\text{T} = (\alpha - \text{T}) - \text{C}\,c - \text{A}\,a. \qquad [41.]$$

Δ T is $\left\{\begin{matrix}+\\-\end{matrix}\right\}$ for chronometer $\left\{\begin{matrix}\text{slow}\\\text{fast}\end{matrix}\right\}$.

An equation similar to this may be formed for each star.

When the greatest accuracy is desired the method of least squares is used for the determination of $\Delta\,\text{T}_o$, the chronometer correction as shown by the whole time set.

If the method of least squares is needed, it may be found fully explained in "Coast and Geodetic Survey Report" for 1880, Appendix 14, and also in Chauvenet's "Astronomy," Vol. II.

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* The factors A B C have been tabulated, and may be found in "Coast and Geodetic Survey Report" for 1880, Appendix 14; also Report of 1874.

For most purposes, however, the method of least squares is too laborious, hence the following abridgment is substituted, the final chronometer correction rarely differing more than $0{\cdot}01^s$ from that found by the method of least squares.

The $(\alpha - T)$'s are found as explained above, and we must now determine the collimation and azimuth corrections.

The object in selecting the stars, as explained on page 163, will now be apparent. In order to find c, we have observed several stars near the zenith, both with telescope direct and with it reversed. Hence the mean of those before reversal must differ from the mean of those after reversal by twice the value of c multiplied by the mean of the collimation factors, C, of the stars observed.

Let $\frac{[\alpha - T]}{n}$ and $\frac{[\alpha - T]_W}{n_1}$ represent the means of the time stars for telescope direct and reversed respectively, n and n_1 being the number of time stars observed in each case, and $+\frac{[C]}{n}$ and $-\frac{[C]_W}{n_1}$ the mean of corresponding collimation factors. They must have opposite signs. Then we find c from

$$c = \frac{\frac{[\alpha - T]}{n} - \frac{[\alpha - T]_W}{n_1}}{\frac{[C]}{n} - \frac{[C]_W}{n_1}} \qquad [42.]$$

If the star list is perfect, as explained on page 163, the value for c thus found is the one required, but it is practically impossible to select a perfect list of stars without extending the time so greatly as to introduce unknown errors due to changes in the instrument, etc.; hence we may have to correct c slightly after finding the first values for the a's. With the value of c as found from [42] we obtain the collimation corrections to be applied to $\frac{[\alpha - T]}{n}$, $\frac{[\alpha - T]_W}{n_1}$, and also to both the $(\alpha - T)$'s of the stars observed for the azimuthal deviation. A star is observed for this purpose both before the reversal and afterward, as the azimuth is rarely the same in both cases, and it is best to compute each separately.

We have now four values for $\alpha - T - Cc$ which we must use in computing a_E and a_W, the azimuthal deviations before and after

reversal. We take the mean of the azimuth factors A in the same way as we did the C factors.

Let $\frac{[a-T]}{n} - \frac{[C]}{n}c$ and $\frac{[a-T]_W}{n_1} - \frac{[C]_W}{n_1}c$ represent the values of the two time-star means after applying the collimation correction, and $(a - T - C\,c)_E$ and $(a - T - C\,c)_W$ the two azimuth stars corrected for collimation. We find a_E and a_W, then, from

$$\left.\begin{aligned} a_E &= \frac{\left(\frac{[a-T]}{n} - \frac{[C]}{n}c\right) - (a - T - C\,c)_E}{\frac{[A]}{n} - A_E} \quad \cdots\cdot \\ a_W &= \frac{\left(\frac{[a-T]_W}{n_1} - \frac{[C]_W}{n_1}\right) - (a - T - C\,c)_W}{\frac{[A]_W}{n_1} - A_W} \end{aligned}\right\} \qquad [43.]$$

Applying the corrections for azimuth now, we should get identically the same results for the $(a - T - C\,c - A\,a)$'s for the four quantities. If the two values for telescope direct (or for telescope reversed) differ, an error has been made in determining the azimuth correction. If the values for telescope direct differ from those for telescope reversed the value of c is erroneous, and we must correct it by a quantity c_1, such that

$$c_1 = \frac{\left(\frac{[a-T]}{n} - \frac{[C]}{n}c - \frac{[A]}{n}a_E\right) - \left(\frac{[a-T]_W}{n_1} - \frac{[C]_W}{n_1} - \frac{[A]_W}{n_1}a_W\right)}{\frac{[C]}{n} - \frac{[C]_W}{n_1}} \qquad [44.]$$

and then with this new value for c obtain new values for the $(a - T - C\,c)$'s, and with these values redetermine a_E and a_W. After applying these second values we will nearly always make the $(a - T - C\,c - A\,a)$'s, E and W, agree within $0{\cdot}01^s$, which is sufficiently accurate.

With these values for c and a we obtain the value of $\Delta\,T$ for each star, and $\Delta\,T_0$ by taking the mean of the $\Delta\,T$'s.

In order to obtain a rough value of the probable error of $\Delta\,T_0$ we determine the residuals (v) by taking the difference between each $\Delta\,T_t$ and $\Delta\,T_a$ for both positions of the telescope, $\Delta\,T_a$ being the $\Delta\,T$ for the azimuth star, and $\Delta\,T_t$ the $\Delta\,T$ for each time star.

The probable error, ϵ, of $\Delta\,T_0$ is then found from

$$\epsilon = \pm\, 0{\cdot}84\, \frac{[v]}{(n-1)\sqrt{n}}, \text{ or } \epsilon = \pm \sqrt{\frac{0{\cdot}455\,[v^2]}{(n-1)\,n}} \qquad [45.]$$

In getting $[v]$ the signs are disregarded.

As a final check upon the work the sum of the + residuals should equal the sum of the − in each position of the telescope, within $0{\cdot}02^s$, and the mean of the $A\,a$'s and $C\,c$'s for the time stars of each group should be equal to the respective values found for the mean of the time stars when first determining the values of c and a.

This computation is a little weak, from the fact that all stars are given equal weight, but experience has shown that weighting does not affect the result very materially, rarely giving a result differing more than $0{\cdot}01^s$ from that obtained by this method when the time set is well selected and the observations made by an experienced observer.

Example. Computation of chronometer correction from the observations on page 165.

Station, Lafayette Park. Date, November 25, 1896. Observer, O. B. F. Computer, O. B. F.

STAR.	Clamp.	a−T	C.	A.	Cc	Aa.	ΔT.	v	
γ′ Androm...	W.	+29·11	−1·84	− ·09	− ·87^5	+ ·08^5	+29·45	− ·05^5	[v]=0·285
β Trianguli...	W.	+29·19	−1·21	+ ·07	− ·84	− ·02^5	·55^5	+ ·05	
γ Trianguli...	W.	+29·13	−1·20	+ ·09	− ·28^5	− ·08^5	·50	− ·00^5	Prob. error=
ξ² Ceti.......	W.	+29·03	−1·01	+ ·50	− ·28^5	− ·19^5	·51	+ ·00^5	$\pm 0{\cdot}84 \frac{[v]}{(n-1)\sqrt{n}}=$
36 Cassiop ...	W.	+29·81	−3·80	−1·87	− ·92^5	+ ·78	·50^5		
ν Arietis.....	E.	+29·70	+1·07	+ ·80	+ ·80	− ·12^5	·52^5	+ ·02	$\pm 0{\cdot}84 \frac{0{\cdot}285}{9\sqrt{10}}=$
μ Ceti.	E.	+29·54	+1·01	+ ·48	+ ·28^5	− ·20	·45^5	− ·05	
41 Arietis....	E.	+29·78	+1·12	+ ·21	+ ·81^5	− ·09	·55^5	+ ·05	±0·007
47 Cephei....	E.	+32·41	+5·24	−3·45	+1·46^5	+1·44	·50^5		
ρ Persei......	E.	+29·87	+1·28	− ·01	+ ·36	+ ·00^5	·50^5	− ·00	

2D APPROXIMATION.

						a−T−Cc.	Aa.	a−T−Cc−Aa.	
Time stars...	W.	29·11^5	−1·19	+ ·14	− ·88^5	+29·45	− ·05^5	+29·50^5	
Az. star		29·81	−3·80	−1·87	− ·92^5	+30·28^5	+ ·78	+29·50^5	$c = +0{\cdot}280$
									$a_w = -0{\cdot}390$
Time stars...	E.	29·72	+1·12	+ ·24^5	+ ·81^5	+29·40^5	− ·10	+29·50^5	$a_e = -0{\cdot}417$
Az. star......		32·41	+5·24	−3·45	+1·46^5	+3094·^{5}	+1·44	+29·50^5	

At $2^h\,29^m\,04^s\ \Delta T_0 = +\,29{\cdot}506^s \pm 0{\cdot}007$, or chronometer, Hutton No. 211, is slow $29{\cdot}506^s$ at true sidereal time $2^h\,29^m\,04^s$.

1ST APPROXIMATION.

	a−T.	C.	A.	Cc.	a−T−Cc.	Aa.	a−T−Cc−Aa.	
W.	[+29·11^5]	[−1·19]	[+0·14]	− ·81	+29·42^5	− ·05	29·47^5	
	+29·81	−3·80	−1·87	− ·86	+30·17	+ ·69^5	29·47^5	$c = +0{\cdot}260$
								$a_w = -0{\cdot}371$
E.	[+29·72]	[+1·12]	[+0·24]	+ ·29	+29·43	− ·10^5	29·58^5	$a_e = -0{\cdot}488$
	+32·41	+5·24	−3·45	+1·86	+31·05	+1·51^5	29·58^5	

$$c = \frac{29{\cdot}11^5 - 29{\cdot}72}{-1{\cdot}19 - (+1{\cdot}12)} = +0{\cdot}260,$$

$$a_W = \frac{29{\cdot}42^5 - 30{\cdot}17}{+0{\cdot}14 - (-1{\cdot}87)} = -0{\cdot}371,\ a_E = \frac{29{\cdot}43 - 31{\cdot}05}{+0{\cdot}24^5 - (-3{\cdot}45)} = -0{\cdot}438$$

$$c_1 = \frac{29{\cdot}47^5 - 29{\cdot}53^5}{-1{\cdot}19 - (+1{\cdot}12)} = +0{\cdot}026.$$

By examining the $A\,a$'s we see that the second half is much more sensitive to a change in c than the first; hence we do not use the whole of c_1, but merely $c_1 = +0{\cdot}020$, and making the computation with this new value for c obtain $\alpha - T - C\,c - A\,a$, identical for each group.

LONGITUDE.

699. The longitude of a point on the earth's surface is the angle at the pole included between the meridian of the point and some assumed meridian called *first* or *initial meridian.*

The difference in longitude between two points is the angle included by their meridians.

The difference in longitude of two points we know is fifteen times the difference in time between them; hence by comparing the local times at two points we can find their difference in longitude.

If $\Delta\lambda$ = difference in longitude between two points, T and T_o the local times of the two points (both sidereal or both solar), we obtain $\Delta\lambda = 15\ (T_o - T)$.

If T_o is at the initial meridian, then we have the longitude of the second station (λ) from $\lambda = 15\ (T_o - T)$.

The determination of a longitude astronomically is therefore the determination of the difference in time between the two points at the same instant.

The methods of determining the time at each point is the same as explained on page 150 *et seq.*, and we will now consider the methods for comparing these times.

There are various methods for obtaining differences of longitude, but only three will be considered here—viz., by the transportation of chronometers, by terrestrial or celestial signals, and by telegraphic signals. If it is desired to obtain the longitude in some other way, full descriptions may be found in works on practical astronomy.

The most accurate and usually the most economical method is the telegraphic, and should be used whenever possible. In regions where there are no telegraphic connections the most accurate method of determination is by transporting chronometers, particularly when the two points are on the coast and the chronometers can be transported by ship. If the two points are inland the method of terrestrial signals is probably best.

700. By Transporting Chronometers. A number of chronometers are placed near the center of motion of a ship, allowed to swing freely in their gimbals, kept at as uniform a temperature as possible, wound regularly, and every care taken not to disturb them in any way. The number of chronometers will depend upon the length of the trip, character of the chronometers, and accuracy desirable. For recent work in Alaska, nine chronometers were used in the southeastern part, where the water was smooth, and twenty-one for the work between Sitka and the Aleutian Islands. The ship carrying the chronometers will start from the first station just after observations for time have been obtained, one of the station chronometers being carried on board and compared with the traveling chronometers just before starting.

There should be both mean-time and sidereal chronometers at all places where comparisons are desired, so that they can be made by coincidences. Since there are nearly four minutes more in a sidereal day than in a mean solar day, the sidereal chronometer gains a second on the mean time about every six minutes, or, as the chronometers beat half seconds, a mean-time and sidereal chronometer will beat in unison once in every three minutes. By noting the times on each chronometer when they beat together we can get a very accurate comparison, and this is the method always used when accuracy is desired.

The ship then proceeds to the other station as rapidly as possible, and one of the station chronometers there is compared with those on board as soon as the ship arrives, and again just before she is ready to leave, if she remains more than a few hours.

Three or more chronometers are usually kept at each station, only one being disturbed in any way excepting to wind them. The object in having more than one is to give a better determination of the local time at the epoch of the comparisons on the ship, when

the observations for time are not very close to the comparison on the ship and the rates of the station chronometers have to be depended upon for this interval. The chronometer which is carried on board the ship (called the *hack*) is compared with the other station chronometers immediately before and immediately after the comparisons on board the ship, and the mean of the corrections for all of them used to determine the local times of the comparisons on the ship. The observations are made to determine the correction for the hack, which is immediately compared with the other station chronometers, thus enabling us to determine their errors at that time.

Having now determined the local time at each station at the instant of comparing with a traveling chronometer, let ΔT_e and ΔT_w be the corrections to the traveling chronometer shown at eastern and western stations respectively at the epochs T_e and T_w shown by the chronometer face.

Let R = rate per day of the traveling chronometer as shown by the chronometer;

λ = difference in longitude;

then $T_w + \Delta T_w$ is the true time at the western station at the time of comparison (T_w); and the corresponding time at the eastern station, considering the rate of the chronometer uniform, is

$$T_e + (T_w - T_e) + R\,(T_w - T_e) + \Delta T_e$$

hence, subtracting the second from the first, we have

$$\lambda = \Delta T_w - [\Delta T_e + R\,(T_w - T_e)] \qquad [46.]$$

We thus see that the longitude difference is expressed as the difference in the chronometer corrections, the absolute indications of the chronometer being of no value except so far as they are necessary in finding the interval with which the rate is computed.

To determine the rate of the traveling chronometer, we take the corrections obtained at the first station at the time of leaving and arrival after making the trip, eliminating from this interval the time the ship was stopping at the other station, thus getting only the traveling rate, since the rate while the ship is in motion is nearly always different from that while she is lying still.

If T_e, T_w, T'_w, T'_e = time of leaving E, arriving at W, leaving W, and arriving at E, respectively, as shown by the chronometer,

$\Delta_e, \Delta_w, \Delta'_w, \Delta'_e$ = corresponding corrections obtained from the observations;

Then $T_w - T_e + T'_e - T'_w$ = time during which chronometer was traveling; and $(\Delta'_e - \Delta_e) - (\Delta'_w - \Delta_w)$ = corresponding change in the chronometer. Hence the daily rate, R, is found from

$$R = \frac{(\Delta'_e - \Delta_e) - (\Delta'_w - \Delta_w)}{T_w - T_e + T'_e - T'_w} \qquad [47.]$$

the T's being given in days and fractions of a day. Each chronometer therefore furnishes a result for the difference of longitude for each round trip.

If the chronometer rate were uniform during the time the ship was under way this would be as good as we could get; but as the rate is always a very uncertain quantity, particularly at sea, it is best to make a second set of determinations, obtaining the rate by taking the interval from the time of leaving the second station to the return thereto, deducting the accumulated rate while at the first station. In the first case we will have n determinations of the difference in longitude for each chronometer (n being the number of round trips), and in the second case $n - 1$, with two single trips or half trips, which may be combined and called one round trip; thus we have $2\,n$ results for each chronometer. The combination weights (p) are obtained from

$$p = \frac{n - 1}{[(l_o - l_n)^2]} \qquad [48.]$$

where l_o represents the mean of all the results by a chronometer, l_n any one of them, and $[(l_o - l_n)^2]$ the sum of the squares of all the residuals for this chronometer. p is the reciprocal of the square of the theoretical mean error of the chronometer.

Having obtained the weights, p, we take the weighted mean of all the chronometers for each trip and thus obtain as many results as there are trips—i. e., $2\,n$. To combine these results, they may be weighted in the inverse proportion to the duration of the trip, and if the rates of the station chronometer had to be depended upon for some time before obtaining observations, this interval may be added to that of the length of the trip.

701. By Exchange of Terrestrial Signals. Since we merely want the difference in time between the two stations whose difference in

longitude is desired, we may get it by noting at each station the instant at which a signal was made. If no point can be obtained from which both stations can be seen, we may use several points, exchanging signals from the first to the second, second to the third, and so on to the other station. An accurate timepiece must be used at each point of exchange, so as to get the difference between the time of receiving and forwarding a signal.

702. By Celestial Signals. Certain celestial phenomena which are visible at the same absolute instant from various parts of the globe may be used instead of terrestrial signals. Among these may be noted:

1. Instant of beginning and ending of an eclipse of the moon. This can only be used for a rough determination, as it is impossible to distinguish the exact instant owing to the imperfect definition of the earth's shadow.

2. Eclipses of Jupiter's satellites by the shadow of that planet. As the times of the occurrence of these phenomena are usually given in the ephemerides, the absolute longitude may be obtained directly by taking the difference between the observed local time of the eclipse and the computed time of the ephemeris. Both the disappearance and appearance should be noted, in order to eliminate the effect of varying telescopic powers.

3. Occultations of Jupiter's satellites by the body of the planet.

4. Transits of the satellites over Jupiter's disk.

Other celestial phenomena may be used, but are not very accurate and are rarely used.

703. By Telegraphic Signals. In this method the local times at the two stations are compared by sending signals over an electro-telegraphic wire which connects the two stations.

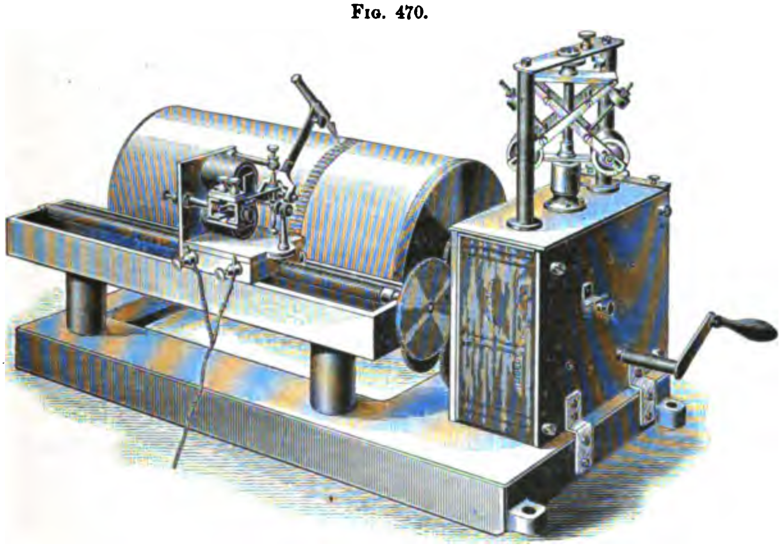
For this work a transit instrument is mounted at each station and time sets observed, as described on page 156 and following. Two time sets are observed at each station, so as to get the chronometer rate for a short period to determine its error at the instant of sending or receiving the signals. The time of exchanging signals should be in the interval between the epochs of the two time sets, and the two time sets are usually taken as close together as observers

can work accurately. At least ten stars should be observed in each set, thus making twenty stars necessary for a complete determination at each station.

In order to eliminate the errors in the star places both observers should use the same stars. This, however, limits the east and west distance between the stations, as, in order to bring both exchanges between the epochs of the time sets, the two stations can not be much more than an hour apart, unless an interval is introduced between the time sets.

704. The Chronograph. The chronograph (Fig. 470) is always used for these observations, the observer breaking the circuit by means of a small key, which he holds in his hand, at the instant

Fig. 470.



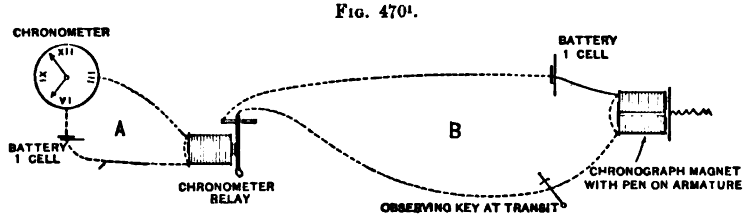
the star crosses each line of the diaphragm of the telescope. This break is recorded on the paper which is wrapped around the revolving cylinder of the chronograph. The observer's key and the chronometer are placed on the same circuit and connected with an electro-magnet whose armature carries the pen which records on the cylinder of the chronograph (Fig. 470[1]). The chronometer is made to break this circuit every second or every two seconds, and,

in order to distinguish the minute, the chronometer does not break the zero second if it breaks seconds. If it breaks every two seconds it breaks the fifty-ninth (or first) as well as all the even seconds. The chronograph should be regulated as uniformly as possible, so that the intervals between the chronometer breaks will be uniform.

The breaks that are made for the transit of the star are read off to the nearest half-tenth of a second by means of a glass scale.

The exchange of signals is usually recorded upon the same sheets with the time observations, by switching both chronographs

FIG. 470[1].



ARRANGEMENT OF CIRCUITS WHILE OBSERVING TRANSITS

into the main line connecting the two stations. Each observer makes about thirty arbitrary breaks with the talking key on his switch board, and each break is recorded on both chronographs. One arbitrary break is usually made in each of the two-second intervals, so that each observer sends for about a minute, and the telegraph line is needed only about two minutes for the whole exchange. The breaks are made at both ends, in order to determine and eliminate the time of transmission of the electric effect, which includes the retardation of the current through the wire, armature and induction time, etc. While exchanging signals the chronograph is run at double speed, so that the arbitrary signals may be read to hundredths of seconds.

For the computation of the difference of longitude we have the following simple formulas:

Let t_e represent the chronometer time of sending a signal at the eastern station;

t_w, the chronometer time at the western station when receiving this signal; and

t'_e, t'_w, the corresponding times for a signal sent by the western station.

μ = time of transmission, supposed to be the same both ways;

λ = difference in longitude, west being reckoned positive;

$\Delta\, t_e$, $\Delta\, t'_e$, $\Delta\, t_w$, and $\Delta\, t'_w$ be the chronometer corrections at the epochs t_e, t'_e, t_w, and t'_w, respectively;

Then we have from an eastern signal,

$$\lambda_e = \lambda - \mu = t_e + \Delta\, t_e - (t_w + \Delta\, t_w) \qquad [49.]$$

and from a western signal,

$$\lambda_w = \lambda + \mu = t'_e + \Delta\, t'_e - (t'_w + \Delta\, t'_w) \qquad [50.]$$

whence $$\lambda = \tfrac{1}{2}\,(\lambda_e + \lambda_w) \text{ and } \mu = \tfrac{1}{2}\,(\lambda_w - \lambda_e) \qquad [51.]$$

In order to simplify the computation and also be certain to use only such signals as appear on both chronographs, we obtain the differences $t_e - t_w$ and $t'_e - t'_w$ for each signal, and then use their mean values in place of the values given in the formulas, as we can obtain the chronometer corrections for the mean of the t_e's, t_w's, etc., and thus compute each only once instead of about thirty times.

705. Personal Equation. This difference in longitude requires a further correction for the personal equation of the two observers. The usual procedure is for the observers to exchange stations after completing half the observations, and as it is usually considered necessary to get observations on ten nights for primary work, each observer will make five nights' observations at each station. If their personal equation is constant this exchange entirely eliminates it from the resulting mean value of λ. The personal equation is usually somewhat variable, and this, together with atmospheric and other local disturbances, make it desirable to get a number of determinations on different nights. The personal equation may change as much as 1^s in a few years, even with experienced observers. The great astronomers Bessel and Struve had no personal equation in 1814, $0{\cdot}8^s$ in 1821, and 1 in 1823, thus indicating a gradual change.

If it is impracticable to exchange observers, their personal equation should be observed just before beginning the longitude work and again immediately after, if possible. The conditions while observing personal equation should be as nearly like those under

which the observations for longitude work were conducted—same instruments, chronographs, batteries, observing keys, etc.

The usual method is for both observers to tap the transits of the same stars, one over the first half and the other over the second half of the lines of the diaphragm, each beginning alternately so as to eliminate any effect due to errors in the intervals of the lines. Each is reduced to the mean thread by the known equatorial intervals of the threads, and the difference must be their personal equation.

Another method, and perhaps a more reliable one, is for each observer to use his own instrument, both being mounted close together and in the same meridian, and observe the same complete time set to determine the error of the same chronometer; or let each one use his own chronometer and exchange signals in the midst of the observations just the same as when observing the longitude. This is practically observing the difference in longitude between two points which we know is zero, and therefore any difference developed must be due to personal equation.

In combining the results for difference of longitude they may be assigned relative weights p, such that

$$p = \frac{p_1 p_2}{p_1 + p_2} \qquad [52.]$$

where p_1 and p_2 are the reciprocals of the squares of the mean errors of the time determinations at the two stations respectively.

706. Adjustment. Whenever we get the difference in longitude between two points by different routes we will usually get a small discrepancy. When we have a number of loops we can apply the method of least squares to adjust the various discrepancies. The method of procedure is practically the same as the method used in adjusting differences in height, explained on page 240. The weights, however, will be derived from the probable errors of the various lines that enter the loop.

LATITUDE.

707. Astronomical Latitude. The astronomical latitude of a place is the angle which the vertical at that place makes with the plane of the equator. It is equal to the altitude of the elevated pole

at that place. The latitude of a place is represented by ϕ, the zenith distance of a star (or celestial body) on the meridian of the place by z, and δ is its declination.

Then $\phi = \delta \pm z$, + when the star is between the zenith and equator, and − when beyond the zenith. δ must be counted from the equator through the zenith, and over the pole if the star is a subpolar.

708. First Method. By Meridian Altitudes or Zenith Distances. When the instrument is perfectly adjusted in the meridian we may measure the altitude (h) or zenith distance (z) of a star, and obtain the latitude by substituting z in the above formula, δ being obtained from an ephemeris. If the instrument is not adjusted in the meridian, a star may be followed until it reaches its greatest h, which is the meridian altitude. With the bodies of the solar system the declination changes so rapidly that the greatest altitude may be reached either before or after the meridian passage; hence, if the greatest altitude is observed, a correction must be applied in order to obtain the meridian altitude.

By observing two stars the mean of whose δ's are about equal to the zenith, and which culminate about the same time, we will obtain a latitude almost free from the effect of refraction, and also any constant errors of the instrument peculiar to that zenith distance.

Example:

	α Aquilæ.	α Cephei.	
True $z =$	$+ 26° \; 34' \; 27{\cdot}5''$	$- 26° \; 54' \; 28{\cdot}3''$	Corrected for refrac.
$\delta =$	$+ \; 8 \; \; 29 \; \; 22{\cdot}7$	$+ 61 \; \; 58 \; \; 21{\cdot}1$	
$\phi =$	$+ 35° \; 03' \; 50{\cdot}2''$	$+ 35° \; 03' \; 52{\cdot}8''$	
Mean =	$+ 35° \; 03' \; 51{\cdot}5''$		

709. Second Method. By Altitude of a Star at Upper and Lower Culmination. Whenever the polar distance of a star is less than the altitude of the elevated pole it may be observed at both its upper and lower culminations.

Let h = true altitude at upper culmination;
h_1 = true altitude at lower culmination;
p = star's polar distance at upper culmination;
p_1 = star's polar distance at lower culmination;

we have $\phi = h - p$

$$\phi = h_1 + p_1$$

the mean of which is

$$\phi = \tfrac{1}{2}(h + h_1) + \tfrac{1}{2}(p_1 - p) \qquad [53.]$$

hence the absolute values of p and p_1 are not required, but only their difference, $p_1 - p$. The change of a star's declination in twelve hours may usually be neglected, and we have the simple formula

$$\phi = \tfrac{1}{2}(h + h_1) \qquad [54.]$$

In fixed observatories this method is used very much, as the latitude is free from any errors in the declination of the star, $p_1 - p$ being obtained very accurately, as it depends upon precession and nutation.

This method is still subject to the whole error in the refraction, which, however, will usually be very small.

710. Third Method. By Single Altitude at a Given Time. Note the time by the chronometer at the instant at which the altitude is observed. The chronometer correction being known, we find the true local time, and then the star's hour angle (t) by the formula

$$t = T_s - \alpha$$

where t_s is the local sidereal time of the observation, and α the star's right ascension; t is the apparent solar time if the sun is observed. In equation [1], page 139, we have

$$\sin. h = \sin. \phi \sin. \delta + \cos. \phi \cos. \delta \cos. t \qquad [55.]$$

in which ϕ is the only unknown quantity. To determine it, assume d and D to satisfy the conditions

$$d \sin. D = \sin. \delta$$

$$d \cos. D = \cos. \delta \cos. t$$

then [1] becomes

$$d \cos. (\phi - D) = \sin. h \qquad [56.]$$

For practical convenience, put

$$\phi - D = \pm \gamma$$

Then, by eliminating d, the solution may be put under the form

$$\tan. D = \tan. \delta \sec. t$$

$$\cos. \gamma = \sin. h \sin. D \operatorname{cosec.} \delta$$

$$\phi = D \pm \gamma \qquad [57.]$$

D may always be taken less than 90°, and with same sign as tan. δ, since t should always be less than 90°, or 6^h.

Two values for ϕ will be found, since the sign of γ is indeterminate, but the actual position of the observer will be known with sufficient accuracy to determine which value should be taken, excepting in some peculiar cases at sea. ϕ can not exceed 90°.

Example. March 27, 1856, assumed $\phi = 23°$ S.; longitude $= 43°\ 14'$ W. ($2^h\ 52^m\ 56^s$); the double altitude of the sun's lower limb ($\underline{\odot}$), observed with sextant and artificial horizon, was $114°40'30''$ at $4^h\ 21^m\ 15^s$ by a chronometer which was fast of Greenwich time $2^m\ 30^s$. Index correction of sextant $= -1'\ 12''$. Barometer, 29·72 inches. Attached thermometer $= 61°$ F.; ext. therm. $= 61°$ F. Required true ϕ.

	° ′ ″
Sextant reading =	114 40 30
Index correction =	− 1 12
	114 39 18
App. alt. $\underline{\odot}$ =	57 19 39
Semidiameter =	+ 16 3
Ref. and par. =	− 31
h =	57 35 11
δ =	+ 2 51 30

	h m s
Chronometer	4 21 15
Correction, Δ T =	− 2 30
Gr. date, March 27,	4 18 45
Longitude =	2 52 56
Local mean time =	1 25 49
Equation of time =	− 5 19
App. time, t =	1 20 30
	= 20° 7′ 30″

log. sec. $t = 0·027360$
log. tan. $\delta = 8·698351$
log. tan. D $= 8·725711$

$\mathrm{D} = +3°\ 2'\ 38''$
$\gamma = 25\ 58\ 49$
$\mathrm{D} - \gamma = \phi = -22\ 56\ 11$

log. cosec. $\delta = 1·302190$
log. sin. D $= 8·725098$
log. sin. $h = 9·926445$
log. cos. $\gamma = 9·953733$

If equal altitudes are observed as for time, page 151, the latitude may be obtained without knowing the error of the chronometer, the hour angle t being obtained by taking half the elapsed time between the two observations. The chronometer error is not necessary, but its rate should be known, as it affects the elapsed interval. Errors in altitude and time will have the least effect when the observation is in the meridian, and greatest when in the prime vertical. Errors of altitude may be partially eliminated by

taking the mean of results found from stars near the meridian both north and south of zenith, and *constant* errors of the instrument wholly eliminated. In order to reduce the effect of an error in the declination at the same time with that of errors of altitude and time, we should select a star as near the pole as possible and observe it at or near its greatest elongation on either side of the meridian. If the star is near the pole, however, we may effect the reduction as in the sixth method.

When several observations in succession are near the meridian it is best to reduce each separately rather than take the means, unless a correction for second differences is made. The fourth method is well adapted for such computations.

711. Fourth Method. To reduce an Altitude observed at a given Time to the Meridian. If in formula [1] we substitute

$$\cos. t = 1 - 2 \sin.^2 \tfrac{1}{2} t$$

we have

$$\sin. \phi \sin. \delta + \cos. \phi \cos. \delta - 2 \cos. \phi \cos. \delta \sin.^2 \tfrac{1}{2} t = \sin. h$$

But

$$\sin. \phi \sin. \delta + \cos. \phi \cos. \delta = \cos. (\phi \sim \delta)$$

Hence, if we put $z_1 = (\phi \sim \delta)$ (which is the meridian zenith distance for a star whose δ does not change in the short interval), we have

$$\cos. z_1 = \sin. h + \cos. \phi \cos. \delta \, (2 \sin.^2 \tfrac{1}{2} t) \qquad [58.]$$

A rough determination for ϕ will enable us to get a close approximation if the observations are not too far from the meridian, and a second computation nearly always gives z_1 with the required degree of accuracy.

Having obtained z_1, we get ϕ the same as in the first method.

If δ changes, this method still holds if we take the value of δ for the instant of the observation. This method is only convenient when the computer has a table of natural sines and cosines, as well as logarithms, and the logarithmic values of $(2 \sin.^2 \frac{1}{2} t)$.

Example. Same as example on page 187. $h = 57° \; 35' \; 11''$, $\delta = +2° \; 51' \; 30''$, $t = 1^h \; 20^m \; 30^s$. Approximate value of $\phi = -23°$.

$$\log. (2 \sin.^2 \tfrac{1}{2} t) = 8{\cdot}785726$$

$$\log. \cos. \phi \qquad = 9{\cdot}964026$$

nat. sin. $h = 0{\cdot}844201$		log. cos. δ	$= 9{\cdot}999459$
nat. no. $= 0{\cdot}056132$	. . . .	log.	$= 8{\cdot}749211$
nat. cos. $z_1 = 0{\cdot}900333$			

$$z_1 = -25^\circ\, 47'\, 54'' \text{ (zenith south of sun)}$$
$$\delta = + \quad 2 \quad 51 \quad 30$$
$$\phi = -22^\circ\, 56'\, 24''$$

differing but 13″ from the true value, although the assumed latitude was in error nearly 4′. Repeating the computation with $-22^\circ\ 56'\ 24''$ as the approximate latitude, we find $\phi = -22^\circ\ 56'\ 11''$, exactly as in the third method.

712. Fifth Method. By Reduction of Circummeridian Altitudes. When a number of altitudes are observed very near the meridian they are called circummeridian altitudes. Each circummeridian altitude reduced to the meridian gives nearly as accurate a result as if the observations were taken on the meridian.

If in equation [58] we substitute

$$h_1 = \text{meridian altitude} = 90^\circ - z_1$$

we have $\quad \sin. h_1 - \sin. h = 2 \cos. \phi \cos. \delta \sin.^2 \tfrac{1}{2} t$

But $\quad \sin. h_1 - \sin. h = 2 \cos. \tfrac{1}{2} (h_1 + h) \sin. \tfrac{1}{2} (h_1 - h)$

and hence

$$\sin. \tfrac{1}{2} (h_1 - h) = \frac{\cos. \phi \cos. \delta \sin.^2 \tfrac{1}{2} t}{\cos. \tfrac{1}{2} (h_1 + h)} \qquad [59.]$$

This equation may be used to obtain the correction $(h_1 - h)$ to h necessary to reduce it to a meridian altitude, but it requires an approximate knowledge of both ϕ and h, the latter being obtained from the assumed value of ϕ by

$$h_1 = 90^\circ - (\phi \sim \delta)$$

It does not approximate as rapidly as by the last method, but has the advantage of not requiring tables of natural functions.

An approximate method of reducing circummeridian altitudes is found by regarding the small arc $\tfrac{1}{2} (h_1 - h)$ as sensibly equal to its sine—i. e., $\sin. \tfrac{1}{2} (h_1 - h) = \tfrac{1}{2} (h_1 - h) \sin. 1''$, and taking h_1 for $\tfrac{1}{2} (h_1 + h)$, from which it differs very little. Then equation [59] becomes

$$h_1 - h = \frac{\cos. \phi \cos. \delta}{\cos. h_1} \; \frac{2 \sin.^2 \tfrac{1}{2} t}{\sin. 1''} \qquad [60.]$$

The value, in seconds, of

$$\frac{2 \sin.^2 \frac{1}{2} t}{\sin. 1''} = m$$

has been tabulated with the argument t.

Let h' h'', etc., be the observed altitudes corrected for refraction, etc.; t', t'', etc., the hour angles deduced from the observed chronometer times; m', m'', etc., the values of m, from the table, and put the constant factor

$$\frac{\cos. \phi \cos. \delta}{\cos. h_1} = \frac{\cos. \phi \cos. \delta}{\sin. z_1} = \mathrm{A}$$

and we have

$$h_1 = h' + \mathrm{A}\, m'$$
$$h_1 = h'' + \mathrm{A}\, m'', \text{ etc.}$$

and the mean of all these equations gives

$$h_1 = \frac{h' + h'' + h''' \ldots\ldots \text{etc.}}{n} + \mathrm{A}\frac{m' + m'' + m''' \ldots\ldots \text{etc.}}{n} \qquad [61.]$$

in which n is number of observations, or

$$h_1 = h_o + \mathrm{A}\, m_o$$

in which h_o is mean of observed altitudes corrected for refraction, etc., and m_o mean of the values of m.

When h_1 is found, ϕ is obtained as in the first method. It must be remembered that when the sun is observed the mean of the declinations corresponding to the times of observations must be used, and not the δ of the meridian transit. The error due to making the above assumptions increases with the latitude and diminishes with the declination of the body. It is greatest in the zenith. For stars within 10° of the pole and latitudes less than about 50° the above method will give the correct result within a second for hour angles of nearly 30^m. For $\delta = 80°$ and $\phi = 0°$, t may be about an hour before the error becomes $1''$.

Example. United States Naval Academy, June 22, 1849. Circummeridian altitudes of β Ursæ Minoris. Instruments, sextant and artificial horizon. Barometer, 30·81 inches. Att. therm., 65° F.; ext. therm., 64° F.

$$\alpha = \quad 14^h\, 51^m\, 14{\cdot}0^s$$
$$\Delta\, \mathrm{T} = \quad +1 \quad 45{\cdot}7$$

Chronometer time of star's transit $14^h\, 52^m\, 59{\cdot}7^s$

The hour angles in the column t are found by taking the difference

between each observed chronometer time and this chronometer time of transit.

2 Alt.	Chronom.	t.	m.
° ′ ″	h m s	m s	
108 39 40	14 45 47	7 12·7	102·1
39 50	47 1	5 58·7	70·2
40 40	48 54·5	4 5·2	32·8
41 0	51 29·5	1 30·2	4·4
41 0	54 36·5	1 36·8	5·1
40 30	56 22	3 22·3	22·3
40 20	67 43	4 43·3	43·8
40 0	58 47·5	5 47·8	66·0
40 0	15 0 17·5	7 17·8	104·5
39 20	2 10	9 10·3	165·1
Mean 108 40 14			$m_0 =$ 61·63

Ind. corr. $-$ 14 58

108 25 16 Assumed $\phi =$ 38° 59′ 0″

54 12 38 $\delta =$ 74 46 36·9

Refr.	$-$ 42·0	Approx. $z_1 = -$ 35 47 36·9	log. cos. ϕ = 9.8906
Am_0	$+$ 21·5		log. cos. δ = 9·4193
$h_1 =$	54 12 17·5		log. cosec. z_1 = 0·2329
$z_1 =$	$-$ 35 47 42·5		log. A = 9·5428
$\delta =$	74 46 36·9		log. m_0 = 1·7898
$\phi =$	38 58 54·4		log. A m_0 = 1·3326

713. Sixth Method. By the Pole Star. A very accurate method of finding ϕ is to observe the altitude of Polaris at any hour angle when the time is known. The computation may be made as in the third method, but when a number of successive observations are made it is better to use the following method: Equation [1] becomes (substitute p for δ)

$$\sin. h = \sin. \phi \cos. p + \cos. \phi \sin. p \cos. t \qquad [62.]$$

For Polaris p is always small (now less than 1° 15′), and we may develop ϕ in a series of ascending powers of p and then employ as many terms as are necessary to obtain the required degree of precision. The altitude can not differ from ϕ by more than p; hence,

if we put $\phi = h - x$, x will be a small correction of the same order of magnitude as p.

$\sin. \phi = \sin. (h - x) = \sin. h - x \cos. h - \frac{1}{2} x^2 \sin. h + \frac{1}{6} x^3 \cos. h + \text{etc.}$

$\cos. \phi = \cos. (h - x) = \cos. h + x \sin. h - \frac{1}{2} x^2 \cos. h - \frac{1}{6} x^3 \sin. h + \text{etc.}$

$\sin. p = p - \frac{1}{6} p^3 + \text{etc.}$

$\cos. p = 1 - \frac{1}{2} p^2 + \text{etc.}$

Substituting these values in [62], we readily find

$$x = p \cos. t - \tfrac{1}{2} (x^2 - 2xp \cos. t + p^2) \tan. h \qquad [63.]$$
$$+ \tfrac{1}{6} (x^3 - 3x^2 p \cos. t + 3xp^2 - p^3 \cos. t)$$
$$+ \tfrac{1}{24} (x^4 - 4x^3 p \cos. t + 6x^2 p^2 - 4xp^3 \cos. t + p^4) \tan. h - \text{ etc.}$$

For a first approximation we take the first term, then substitute this in the second term and obtain a second approximation, and then a third and fourth approximation in similar manner. As x and p will be expressed in seconds of arc, we must multiply p^2 by sin. 1″, p^3 by $\sin.^2 1''$, p^4 by $\sin.^3 1''$, in order to make the series homogeneous.

Then the expression for ϕ becomes

$$\phi = h - p \cos. t + \tfrac{1}{2} p^2 \sin. 1'' \sin.^2 t \tan. h$$
$$- \tfrac{1}{6} p^3 \sin.^2 1'' \cos. t \sin.^2 t + \tfrac{1}{8} p^4 \sin.^3 1'' \sin.^4 t \tan.^3 h \qquad [64.]$$

from which we can obtain ϕ within 0·01″.

To find ϕ within 1″, we may use the first line only of this equation; or, substituting $[4{\cdot}38454] = \log. [\frac{1}{2} \sin. 1'']$, we have

$$\phi \text{ (within } 1'') = h - p \cos. t + [4{\cdot}38454] p^2 \sin.^2 t \tan. h \qquad [65.]$$

The last page of the "American Ephemeris" contains a table, and an example of computation, which is very simple. As an ephemeris is necessary for an accurate determination of ϕ, it will always be convenient.

The foregoing methods are adapted to any instrument capable of measuring altitudes, particularly the sextant, which may be used when extreme accuracy is not necessary.

ZENITH TELESCOPE.

714. When the latitude is required with great accuracy the zenith telescope is used (Fig. 471). Captain Andrew Talcott, of the United States Corps of Engineers (1834), devised the method of determining the latitude with it. It was adopted by the United

States Coast and Geodetic Survey in 1846, and since then has received several improvements. As now designed it is a very stable, simple, and convenient instrument. There are two sizes used by the Coast and Geodetic Survey. The larger sizes have telescopes of about 114 centimetres (45 inches) focal length, and magnifying powers of about 100 diameters; the smaller, 66 centimetres (26 inches) focal length.

Fig. 471.

The essential characteristics of the zenith telescope are a very delicate level attached to the telescope tube (used to indicate slight changes in inclination of the telescope after it is set for a star) and an eyepiece micrometer. The vertical axis is made long, to insure steadiness of motion in azimuth. The instrument is used in the meridian the same as a transit, and is adjusted therein in a similar manner.

The Talcott method of observing a latitude consists in the measurement of the small difference of zenith distance of two stars culminating on opposite sides of the zenith at nearly the same altitude and not far apart in time. This small difference is measured with a micrometer and is therefore very accurate. The meridian telescope (Fig. 468) has all the characteristics of the zenith telescope, and the description there, excepting the part referring to the frame, may be applied to the zenith telescope.

715. Adjustment of Zenith Telescope. First adjust the striding level as described on page 42, then make the vertical axis perpendicular by means of foot screws, as shown by the bubble of the striding level remaining in the same place while the telescope is moved about in azimuth. If now the bubble is not in the center the horizontal axis is not perpendicular to the vertical axis. Make it so by moving the adjusting screw at its end until the bubble is central.

The collimation of the vertical line of the telescope may be adjusted the same as that of the engineer's transit, if objects are far enough away so that the eccentricity of the telescope is not appreciable; an adjustment of the collimation, however, is not necessary, excepting to have the zero of reference as near the center of the field as convenient. If the instrument is inclosed so that we can see in but one direction, point upon a distant object and then revolve exactly 180° in azimuth and see if the object is again bisected, correcting half of the error by means of lateral adjusting screws on the side of the micrometer box. The other adjustments are made as described for transit and meridian telescope. When the telescope has been set in the meridian stops are clamped to the horizontal circle to define the two positions, so that the telescope may be revolved from one to the other quickly.

716. Selection of Stars. An observing list of pairs of stars must be prepared. (A convenient form is appended). The "Greenwich Ten-Year Catalogue for 1880" is probably the best list for this purpose at present, although the "British Association Catalogue" has more stars. The latter, however, is for 1850, and hence does not include later observations; it is therefore not so reliable as the "Greenwich Ten-Year Catalogue."

No. of Pair.	Catalogue No. (B.A.C.)	Mag.	α	δ	Z. D.	Setting.	N. or S.	Turns.
			h m s	° ′	° ′	° ′		
1	991	5·9	3 06 55	6 16	31 32	31 33	S.	19
	998	6·5	10 07	69 21	31 34			21
..	....	....	....	....	....	....	..	..
25	2300	6·0	6 57 24	52 55	15 08	15 04	N.	25
	2313	5·9	59 03	22 48	15 00			15
26	2330	5·6	7 02 24	16 06	21 42		S.	30
	2362	5·4	07 24	16 20	21 27	21 34	S.	11
27	2369	7·0	10 51	59 19	21 32		N.	17

In order to select a latitude list we must have a rough determination of the latitude, preferably within 1′, although within 4′ or 5′ will not be very objectionable with the smaller instruments. This may be obtained either from maps, a sextant altitude, or with the zenith telescope itself. Having decided upon the local mean time at which we wish to begin the observations, we find the sidereal time corresponding to it, and then select for our first star the one whose right ascension comes the nearest to this sidereal time. We take double the latitude (already roughly known), and if δ is the declination of the first star with which we wish to begin, enter the catalogue with $2\phi - \delta$ to find a star with declination δ' such that it does not differ from $2\phi - \delta$ by more than the amount that the micrometer can measure. This varies from about 15′ to 20′ in the larger telescopes, to 20′ to 30′ in the smaller. In selecting the pairs it is not necessary to go more than 20^m from δ for δ', as that is usually longer than we need wait between the stars of a pair. After forming all the pairs possible, by taking each star in succession and trying to pair it, we select the best observing list we can from these possible pairs. It is not advisable to select pairs less than about 2^m apart, nor pairs where the two stars are less than about 1^m apart, as the instrument and level do not have time to assume a stable position. In making the final selection of the pairs great weight should be given those where the stars are well determined, either by a large number of observations or by several catalogues; select those with smallest difference between δ and δ'—i. e., stars with most nearly equal zenith distances—and at the same time get pairs as close together as can be observed comfortably. If possible, make the sum of the negative micrometer differences equal the positive, so as to eliminate any possible error in the value of the micrometer screw—e. g., if z_1 and z'_1, z_2 and z'_2, z_3 and z'_3, etc., are the zenith distances of the stars in first, second, third, etc., pairs, z_1, z_2, z_3, etc., being the south zenith distances, and z'_1, z'_2, z'_3, etc., the north zenith distances, then

$$(z_1 - z'_1) + (z_2 - z'_2) + (z_3 - z'_3) +, \text{etc., should} = 0.$$

In order to effect this equality ϕ should be known within 1′.

717. Precision. About twelve pairs of stars, if well selected from a modern catalogue and observed on three or four nights,

will usually give the latitude with a probable error of less than 0·1″.

An examination of the local deflection of the vertical (plumb line) at latitude stations shows that the determination of astronomic latitudes with an accuracy of about a quarter of a second is quite sufficient for most schemes of triangulation, being considerably within the limits of the ordinary deflections. A greater precision is necessary in locating state and national boundaries and determining arcs. For the purpose of investigating the variations of latitude the greatest precision possible is required.

This last is of great importance, as it enables us to determine corrections by which a latitude observed at any time may be reduced to the normal latitude. Observations have been made to determine the variation in latitude extending through two years or more, and show that the range is between 0·2″ and 0·7″. According to Dr. S. C. Chandler's researches,* the variation is dependent upon two cycles, which, however, are not constant. At present one is computed to be about 431 days, and the other, dependent on climatic conditions in the two hemispheres, 365¼ days. Although the actual observed values do not quite agree with the values computed from a formula based upon the above-mentioned researches, they are sufficiently close to inspire much confidence in the results deduced so far.

718. Directions for Observing. With instrument mounted, adjusted, and stops set to indicate positions where the telescope revolves in the meridian, set the vertical circle to read the mean zenith distance of the first pair to be observed, and point toward the north, if north star culminates first (or *vice versa* if south star comes to the meridian first), clamp the telescope when the bubble on the latitude level is central, and set the micrometer wire in that part of the field where the star will appear, increasing micrometer readings corresponding to increasing zenith distances when the micrometer head is on top, or between the observer's head and the telescope, and with a right-handed screw.

* Gould's "Astronomical Journal," a series of articles on variation of latitude, vols. xi and xii, 1891–'93.

As the star enters the field bisect it with the micrometer wire, and keep it so bisected until it reaches the middle, when the time is noted. Then the level and micrometer are read and telescope revolved 180° in azimuth to observe the other star of the pair. The micrometer wire is set in that part of the field where the star is expected and observed the same as the first, reading the level and micrometer immediately afterward. In the larger instruments some observers prefer to make two or three independent bisections with the micrometer wire, reading the head and noting the time for each.

After once setting the vertical circle and beginning the observations the accuracy of the result will depend greatly upon whether the relation between the line of collimation of the telescope and the latitude level has been disturbed or not. This relation must be kept constant during the observations on a pair if good work is desired. If the vertical axis is slightly out of the perpendicular, as shown by the level not reading the same after reversal for second star as before, it is better to bring the bubble to read about the same as for the first star, using the tangent screw, which moves the telescope on its axis but does not disturb its relation to the level, for the purpose, as a large level correction is more likely to introduce an error (through changes in the level value due to temperature and the like) than such an adjustment.

719. Determination of Value of One Division of Micrometer. Several methods have been employed for this purpose, but only two are now used to any extent. One requires the turning of the micrometer box 90° about the axis of the telescope, thus introducing a risk of changing the focal adjustments and therefore the value of the screw, since it depends upon the length of the focus. In the other method a circumpolar star is observed through elongation, noting the times of passage of the star over the micrometer wire placed successively before the star for each turn (or fraction of a turn). In the first method the star is noted in the same way as it transits through culmination. The second method requires a reading of the level to correct for any change in inclination of the telescope during the observations, and also a small correction for differential refraction.

If t_e is the star's hour angle at elongation,

$$t_e = \cot.\delta \tan.\phi,$$

and chronometer time of elongation is

$$a - \Delta T \pm t_e \begin{Bmatrix} + \text{ for W.} \\ - \text{ `` E.} \end{Bmatrix} \text{elongation.}$$

In order to find what time to begin the observations so as to bring the middle of the observations at elongation, we have

$$\sin.\tau = n\,v'' \sin.1'' \operatorname{cosec.} p \quad . \quad . \quad . \quad . \quad [66],$$

where τ is the time before elongation that the nth turn from the center must be observed in order to bring the center at the time of elongation. v'' is a rough value for a turn of the screw in seconds of arc.

As the star does not move in a straight line, it appears to have an accelerated or retarded motion. For the small interval that we use it, however, we may deduce a correction which will make the observations the same as though the star appeared to move uniformly.

If τ is the interval of time between an observation and elongation (or culmination),

z'' = number of seconds of arc from elongation (or culmination) measured on the vertical circle (or horizontal).

$z'' = \dfrac{\cos.\delta \sin.(15\,\tau)}{\sin.1''}$, but by expansion

$\sin.(15\tau) = 15\,\tau \sin.1'' - \frac{1}{6}(15\,\tau \sin.1'')^3 + \frac{1}{120}(15\,\tau \sin.1'')^5 +$ etc.

hence

$$z'' = 15 \cos.\delta\,[\tau - \tfrac{1}{6}(15 \sin.1'')^2 \tau^3 + \tfrac{1}{120}(15 \sin.1'')^4 \tau^5]$$

Therefore we obtain for the required correction the term

$$[\tfrac{1}{6}(15 \sin.1'')^2 \tau^3 - \tfrac{1}{120}(15 \sin.1'')^4 \tau^5]$$

which is tabulated below with the argument τ.

τ	TERM.	τ	TERM.	τ	TERM.	τ	TERM.	τ	TERM.	τ	TERM.
m.	s.	m.	s.	m.	s.	m.	s.	m.	s.	m.	s.
6	0·0	16	0·8	26	3·3	36	8·9	46	18·5	56	33·3
7	0·1	17	0·9	27	3·7	37	9·6	47	19·7	57	35·1
8	0·1	18	1·1	28	4·2	38	10·4	48	21·0	58	37·0
9	0·1	19	1·3	29	4·6	39	11·3	49	22·3	59	39·0
10	0·2	20	1·5	30	5·1	40	12·2	50	23·7	60	41·0
11	0·2	21	1·8	31	5·7	41	13·1	51	25·2	61	43·1
12	0·3	22	2·0	32	6·2	42	14·1	52	26·7	62	45·2
13	0·4	23	2·3	33	6·8	43	15·1	53	28·3	63	47·4
14	0·5	24	2·6	34	7·5	44	16·2	54	29·9	64	49·7
15	0·6	25	3·0	35	8·2	45	17·3	55	31·6	65	52·1

This correction is additive to the observed time before either elongation or culmination, and subtractive after.

The level correction is also applied to the observed times, reducing them to what they should be if the telescope did not move.

Let n_o and s_o be the north and south reading of the level bubble at about the mean of the readings made during the set of observations, n and s any other reading, and b the value of 1 division of the level in seconds of arc; then the correction (c) to any observed time, in seconds of time, is

$$c = \pm \left[(n - s) - (n_o - s_o) \right] \frac{b}{30 \cos. \delta}, \left\{ \begin{matrix} + \text{ for West} \\ - \text{ `` East} \end{matrix} \right\} \text{elongation},$$

n and s being the level reading for that observed time. This equation applies when the graduation of the level is numbered from the center toward the ends. If the graduation increases from N. toward the S., use the formula

$$c = \pm \left[(n_o + s_o) - (n + s) \right] \frac{b}{30 \cos. \delta}, \left\{ \begin{matrix} + \text{ for West} \\ - \text{ `` East} \end{matrix} \right\} \text{elongation};$$

if from S. toward N., use

$$c = \pm \left[(n + s) - (n_o + s_o) \right] \frac{b}{30 \cos. \delta}, \left\{ \begin{matrix} + \text{ for West} \\ - \text{ `` East} \end{matrix} \right\} \text{elongation}.$$

After these two corrections have been applied to the observed times of noting, we have in one column the readings of the micrometer, and in another the corresponding times such as would have been observed if the star had moved uniformly in a vertical line (or horizontal line if observed at culmination). Various methods of combination might be adopted for the determination of the value of a turn of the screw. The method by means of least squares is probably the most accurate, but it is too laborious for ordinary use, and does not give results differing materially from those obtained by the following method: We obtain the time for 10 turns by subtracting the first from the eleventh, the second from the twelfth, etc., until the last value observed has been used. We then take the mean of all these values and divide by 10, to get the value for 1 turn. As this result is in seconds of time, we must multiply by 15, and then by cos. δ, in order to get the absolute value in seconds of arc.

This value, when the observation is through elongation, must be corrected for the difference in refraction. This is found by tak-

ing the change per second at the altitude of observation, as found in a table of mean refraction, and multiplying it by the number of seconds in the value of the screw. It is negative for either elongation.

Whenever the rate of the chronometer is appreciable the time for 1 turn should be corrected therefor before changing it to arc. It is negative for a gaining chronometer, and positive for a losing one.

Several sets of observations should be taken, always using different stars, elongations, etc. Some observers use both the elongation and culmination methods, rejecting the results by the latter method, however, if they differ materially from those of the former.

The probable error of a result from a set may be deduced very simply by taking the sum of the squares of the residuals formed by taking the difference between each value for 10 turns and the mean of all, and substituting this sum in the formula:

$$\epsilon \text{ (in } s \text{ for 10 turns)} = \pm \sqrt{\frac{0{\cdot}455\,[\Delta^2]}{n\,(n-1)}}$$

or probable error of 1 turn (in arc) $= \pm\, 1{\cdot}5 \cos.\, \delta \sqrt{\dfrac{0{\cdot}455\,[\Delta^2]}{n\,(n-1)}}$

720. Computation. General Expression for Latitude. Let z and z' be true meridian zenith distances of the south and north stars of a pair respectively, δ and δ' their declinations, ζ and ζ' observed zenith distances, n and s the north and south readings of the level for south star, n' and s' for north star, b the value of one division of level in arc, r and r' their refraction corrections, m and m' their reductions to meridian; then we have

$$\phi = \delta + z, \text{ and } \phi = \delta' - z',$$

or

$$\phi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(z - z').$$

By taking observed zenith distance and applying corrections for level, refraction, and reduction to meridian, we have

$$\phi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(\zeta - \zeta') + \frac{b}{4}[(n + n') - (s + s')] + \tfrac{1}{2}(r - r') + \tfrac{1}{2}(m' - m) \quad [70.]$$

If M and M′ be the readings of the micrometer (in divisions) for the south and north stars respectively (increasing micrometer read-

ings corresponding to increasing zenith distances), and R the value of 1 division, then

$$\tfrac{1}{2}(\zeta - \zeta') = \tfrac{1}{2}(M - M')\,R$$

If the micrometer reads reversed from this, change $M - M'$ to $M' - M$.

721. Correction for Refraction. The difference of refraction for any pair of stars is so small that we can use the tables of mean refraction. For the elevations at which latitude stars are observed the refraction is nearly proportional to the tangent of zenith distance, and the difference of refraction for two stars of a pair will be given by

$$r - r' = 57{\cdot}7'' \sin.(\zeta - \zeta') \sec.^2 \zeta \qquad [71.]$$

The following table is computed giving $r - r'$ with the arguments, difference in zenith distance, and zenith distance. The sign of this correction is always the same as that of the micrometer difference.

½ DIFF. IN ZENITH DISTANCE.	ZENITH DISTANCE.						½ DIFF. IN ZENITH DISTANCE.	ZENITH DISTANCE.					
	0°	10°	20°	25°	30°	35°		0°	10°	20°	25°	30°	35°
′	″	″	″	″	″	″	′	″	″	″	″	″	″
0	·00	·00	·00	·00	·00	·00	6·5	·11	·11	·12	·13	·14	·16
0·5	·01	·01	·01	·01	·01	·01	7	·12	·12	·13	·14	·15	·18
1	·02	·02	·02	·02	·02	·02	7·5	·13	·13	·14	·15	·16	·19
1·5	·03	·03	·03	·03	·03	·03	8	·13	·14	·15	·16	·18	·21
2	·03	·03	·04	·04	·04	·05	8·5	·14	·15	·16	·17	·19	·22
2·5	·04	·04	·05	·05	·05	·06	9	·15	·16	·17	·18	·20	·23
3	·05	·05	·06	·06	·07	·08	9·5	·16	·17	·18	·20	·21	·24
3·5	·06	·06	·07	·07	·08	·09	10	·17	·18	·19	·21	·23	·26
4	·07	·07	·08	·08	·09	·10	10·5	·18	·19	·20	·22	·24	·27
4·5	·08	·08	·09	·09	·10	·11	11	·18	·19	·21	·23	·25	·28
5	·08	·09	·10	·10	·11	·13	11·5	·19	·20	·22	·24	·26	·30
5·5	·09	·10	·10	·11	·12	·14	12	·20	·21	·23	·25	·27	·31
6	·10	·10	·11	·12	·13	·15	12·5	·21	·21	·24	·26	·28	·32

722. Reduction to Meridian. It is occasionally necessary to observe a star off the meridian. This may be done either by turning the telescope and noting the time the star is bisected (near the middle thread), or by bisecting the star on one side of the center and noting the time, the telescope remaining undisturbed. The first is rarely used, as it disturbs the telescope and level too much for accurate work. Knowing the time the star should cross the

meridian, we get the hour angle τ at which it was observed, and then the correction (m) to the zenith distance may be found from

$$\text{(First case)} \qquad m = \frac{2 \sin.^2 \frac{1}{2}\tau}{\sin. 1''} \quad \frac{\cos. \phi \cos. \delta}{\sin. \tau} \qquad [72.]$$

$$\text{(Second case)} \qquad m = \frac{2 \sin.^2 \frac{1}{2}\tau}{\sin. 1''} \quad \sin. \delta \cos. \delta \qquad [73.]$$

The correction to ϕ, if both stars are observed off the meridian, is $\frac{1}{2}(m' - m)$, and always additive.

The value of m has been computed for the second case when the star is observed off the line of collimation without moving the telescope out of the meridian, for the arguments τ and δ and the correction to ϕ is one half the tabular values.

	10s.	15s.	20s.	25s.	30s.	35s.	40s.	45s.	50s.	55s.	60s.	
δ	"	"	"	"	"	"	"	"	"	"	"	δ
5°	·00	·01	·02	·03	·04	·06	·08	·10	·12	·14	·17	85°
10	·01	·02	·04	·06	·08	·11	·15	·19	·23	·28	·34	80
15	·01	·03	·05	·09	·12	·17	·22	·28	·34	·41	·49	75
20	·02	·04	·07	·11	·16	·22	·28	·36	·44	·53	·63	70
25	·02	·05	·08	·13	·19	·26	·34	·42	·52	·63	·75	65
30	·02	·05	·09	·15	·21	·29	·38	·48	·59	·71	·85	60
35	·03	·06	·10	·16	·23	·31	·41	·53	·64	·77	·92	55
40	·03	·06	·11	·17	·24	·33	·43	·54	·67	·81	·97	50
45	·03	·06	·11	·17	·25	·33	·44	·55	·68	·82	·98	45

723. The following form for computation of the latitude is very convenient. Page 204 is nearly all obtained from the record. The δ's are obtained from the computation of *apparent* places of stars which must be made for each latitude. The *mean star places* used in determining the *apparent* places are obtained from standard catalogues. The following are the best catalogues available at present, the weights assigned each being those of the Coast and Geodetic Survey.

REDUCTION TO APPARENT PLACES OF STARS.

Gr. Ten-Yr. Cat. Star No. 2380. Station, Presidio Ast. Sta.

Year,				Year,			
1896.				1896.	h. m. s.	δ_0=1606	μ'=−0·117
	tan. δ_0	sec. δ_0		α_0=	7 02 24	sin. δ_0	cos. δ_0
	9·4603	0·0174	f		105° 36′		
			+3·25	G. H.		9·4429	9·9826
$\alpha-\alpha_0$	g sin.	h sin.	$\tau\mu$	$\delta-\delta_0$	g cos.	h cos.	i cos. δ_0
	(G+α_0)	(H+α_0)			(G+α_0)	(H+α_0)	$\tau\mu'$
	87° 36′	143° 38′			86° 54′	148° 36′	
Nov. 10	9·9996	9·7730		Nov. 5	8·7330	9·9312	
	1·3488	1·2955			1·3408	1·2923	0·7637
	0·8087	1·0859			0·0738	0·6664	0·7463
	1·1761	1·1761	(=log. 15)	+2″·02	+1·185	−4·638	+5·576
							−0·099
	9·6326	9·9098			87° 36′	143° 38′	
+4s·49	+0·43	+0·81		Nov. 10.	8·6220	9·9059	
					1·3488	1·2955	0·7223
					9·9708	0·6443	0·7049
				+1″·49	+0·935	−4·409	+5·069
							−0·101
					87° 38′	140° 41′	
				Nov. 13	8·6159	9·8885	
					1·3519	1·2974	0·6945
					9 9678	0·6288	0·6771
				+1″·33	+0·929	−4·254	+4·754
							−0·102

α_0 h. m. s.	Date.		h. m. s.	δ_0 ° ′ ″	Date.		° ′ ″
7 02 24	Nov. 10	α=	7 02 28·5	16 05 47·34	Nov. 5	δ=	16 05 49·36
					" 10		48·83
					" 13		48·67

Formulas for Reduction to Apparent Position: Independent Star Numbers.

$$\alpha = \alpha_0 + f + \tau\mu + \tfrac{1}{15} g \sin.(\mathrm{G} + \alpha_0) \tan.\delta_0$$
$$+ \tfrac{1}{15} h \sin.(\mathrm{H} + \alpha_0) \sec.\delta_0 \quad \text{(in time).}$$
$$\delta = \delta_0 + \tau\mu' + g \cos.(\mathrm{G} + \alpha_0)$$
$$+ h \cos.(\mathrm{H} + \alpha_0) \cos.\delta_0 + i \cos.\delta_0 \quad \text{(in arc).}$$

LATITUDE

Presidio Ast. Sta. State, California.

DATE.	CATALOGUE B. A. C.		MICROMETER.		LEVEL.			Meridian distance.	Declination.
	Star No.	N. S.	Reading.	Diff. Z. D.	N.	S.	Diff. N−S.		
1896.			t. d.	t. d.				s.	° ′ ″
Nov. 5	2330	S.	30 40·3		43·7	14·8		154	16 05 49·35
					106·8	85·2			
	2369	N.	15 63·3	+14 77·0	16·7	45·6	−3·8		59 18 31·21
					87·0	108·4	−3·4		
" 6			29 68·8		44·1	15·8			49·24
					107·8	86·4			
			15 46·7	+14 22·1	16·9	45·6	−3·1		31·23
					87·9	109·0	−2·7		
" 10			30 36·3		45·1	16·0			48·82
					120·0	98·8			
			16 16·2	+14 20·1	17·0	46·0	−1·9	25	31·35
					99·0	120·6	−1·8		
" 11			29 76·8		42·4	18·0			48·77
					107·0	84·9			
			15 54·4	+14 22·4	14·7	44·1	−3·4		31·44
					85·9	108·0	−2·0		
" 12			30 22·8		41·4	12·7		32	48·71
					107·0	85·8			
			16 00·0	+14 22·8	14·5	43·2	−3·6		31·54
					86·8	108·0	−2·5		
" 13			30 47·3		48·2	19·7			48·66
					118·5	92·0			
			16 25·2	+14 22·1	21·8	50·5	−4·4	20	31·63
					94·4	115·8	−4·7		

Mr. Farquhar in 1891–'92, when computer on the Coast and Geodetic Survey, computed the mean places for the stars used in the determination of the variation in latitude. He makes the following statement concerning his computations and the catalogues used:

"Combination weights used for adopted polar distances in Rockville latitude list: Twenty catalogues used in computing latitudes on this Survey were tested and their probable errors reported in June, 1890. These results were made the basis of the following series of weights, the unit of weight corresponding to a probable error of $\pm\sqrt{0{\cdot}1''}$ and the ratio of observation error to systematic error being taken (for all catalogues alike) $= \sqrt{5}$, so that if w_∞ be the weight of an infinite number of observations, and w_1 of one, $w = \dfrac{n}{n+5} w_\infty = \dfrac{6n}{n+5} w_1$.

COMPUTATION.

Observer, O. B. F. Instrument, Z. T., No. 3.

Sum and half sum.	CORRECTIONS.				Latitude.	Remarks.	
	Microm.	Level.	Refr.	Merid.			
° ′ ″	° ′ ″	″	″	″	° ′ ″	Δ	$Δ^2$
75 24 20·56						Turned out of meridian.	
87 42 10·28	+5 51·58	−0·75	+0·10	−13·28	87 47 47·88	−·14	·0196
20·47							
10·24	38·46	−0·60	+0·10		48·20	+·18	·0324
20·17							
10·08	37·98	−0·32	+0·10	+ 0·08	47·92	−·10	·0100
20·21							
10·10	38·58	−0·56	+0·10		48·17	+·15	·0225
20·25							
10·12	38·51	−0·68	+0·10	+ 0·07	48·17	+·15	·0225
20·29							
10·14	38·46	−0·95	+0·10	+ 0·05	47·80	−·22	·0484
							·1554=[$Δ^2$]
	n=6		Mean		87 47 48·02		

"The weight of no polar distance, therefore, can exceed six times that of a single observation, given in the table below:

CATALOGUE.	w_1
Lalande δ, Weisse-Bessel ε,	0·015
d'Agelet α, Piazzi β,	·02
Rümker ε,	·025
Taylor β,	·04
Groombridge β,	·06
Armagh '75,	·07
Armagh '40, Jacob β, Smyth β,	·08
Auwers-Bradley α, Paris '45, Main β, Glasgow, Cape '40 ε,	·09
Radcliffe '60,	·10
Radcliffe '45,	·12
Pond β, Cambridge '30 β, Greenwich 6y.	·14
Washington,	·16
Cape '50 ε, Bonn β, Paris '60, Rome,	·18
Paris '75,	·20

CATALOGUE.	w_1
Henderson, Greenwich 7y1, Melb. β, Cape '80 ϵ, Ann Arb. ϵ,	·25
Struve Pos. M. β, Greenwich 12y (1 + 2), Brussels, Becker ϵ,	·3
Pulkowa Merid. Circle, Greenwich 7y2, Cordoba γ,	·35
Abo β, Harv. '85 ϵ,	·4
Greenwich 9y, Harv. '75,	·5
Leiden β,	·6
Romberg a,	·7
Pulkowa Vert. Circle β, Greenwich 10y,	·8

"In this table weights were deduced for catalogues marked

"a . . γ. from probable errors of observation given in the prefaces to the catalogues.

"β . . . . from Boss's investigations, two thirds of his weights being taken—i. e., his unit being supposed to correspond to a probable error of $\pm\sqrt{0{\cdot}15''}$.

"γ . . . . from a determination of systematic error by myself, using Boss's method.

"δ . . . . from determination of observation error by myself and formula above as in (a).

"ϵ . . . . from simple estimate, the places being too few for better methods.

"Others from the results obtained in 1890.

"Piazzi, Taylor, Jacob, Main, and a few others used by Safford, I have to take at second hand, being without the originals. For stars of the 'Berliner Jahrbuch,' Dr. Auwers's combination of the authorities used by him was usually accepted, five eighths of his total weights being allowed them (i. e., his *p. e.* taken $= \pm\sqrt{0{\cdot}16''}$ for $w = 1$), and the weights of this table used for the remaining authorities."

724. Combination of Results. If each pair of stars were observed an equal number of times, and the places for all the stars were equally good, we could take the direct mean of the results and obtain the most probable value for the latitude; but these conditions are rarely fulfilled, hence we must determine the weight to be given each result, and then combine them in proportion to these weights.

Since weights should be inversely proportional to squares of the probable errors (or mean errors), we determine the weight of a result by finding its probable error.

Let e be the probable error of observation for a single determination of the latitude.

ϵ_{δ_1} and ϵ_{δ_2}, the probable errors in the declinations of the two stars of a pair, as found from the catalogues.

n, the number of observations on any pair.

p, the number of pairs.

Δ, the residual obtained by taking the difference between a single determination of a pair and the mean of all the determinations for that pair.

$[\Delta^2]$, the sum of the squares of all the residuals in any one pair; and

$\Sigma[\Delta^2]$, the sum of the $[\Delta^2]$'s for all the pairs;

then

$$e = \pm \sqrt{\frac{0{\cdot}455\,\Sigma[\Delta^2]}{[n]-p}} \qquad [75.]$$

([ ] when used in this way always indicates the summation of all the quantities which the characters inclosed represent.)

This gives us the probable error of any one observation, and hence $\frac{1}{e^2}$ is the weight to be used for each determination of ϕ in combining the various results.

The probable error, ϵ_δ, of a pair due to ϵ_{δ_1} and ϵ_{δ_2} is

$$\epsilon_\delta = \tfrac{1}{2}\sqrt{\epsilon^2_{\delta_1} + \epsilon^2_{\delta_2}} \qquad [76.]$$

hence the probable error (e_ϕ) of the mean result for any one pair is (n being number of determinations of the pair)

$$e^2_\phi = \epsilon^2_\delta + \frac{e^2}{n}$$

and therefore the weight of the mean result for a pair is

$$w = \frac{n}{n\,\epsilon^2_\delta + e^2} = \frac{n}{\frac{n}{4}(\epsilon_{\delta_1} + \epsilon_{\delta_2}) + e^2}$$

or, since weights need only be proportional,

$$w = \frac{n}{n\,(\epsilon^2_{\delta_1} + \epsilon^2_{\delta_2}) + 4\,e^2} \qquad [77.]$$

Having now determined the weight to give each pair in combining them, we obtain the resulting latitude, ϕ_o, from

$$\phi_o = \frac{[w\,\phi]}{[w]} \qquad [78.]$$

with a probable error,

$$\epsilon_{\phi_o} = \pm \sqrt{\frac{0{\cdot}455\,[w\,v^2]}{(p-1)\,[w]}} \qquad [79.]$$

the v^2 being the squares of the residuals found by taking the difference between ϕ_o and the mean result, ϕ, for each pair.

If an equal number of observations are made on each pair and the probable errors of the declinations are equal, then

$$\phi_o = \frac{[\phi]}{p} \qquad [80.]$$

and

$$\epsilon_{\phi_o} = \pm \sqrt{\frac{0{\cdot}455\,[v^2]}{p\,(p-1)}} \qquad [81.]$$

The example on pages 203–205 is one of twenty-seven pairs of stars observed at Presidio Astronomical Station, San Francisco.

As all the observations are of equal weight, we take the arithmetic mean and obtain the residuals (Δ) by taking the difference between this mean and each determination. Taking the sum of the square of the residuals, we have $[\Delta^2] = 0{\cdot}1554$ with $n = 6$, the number of determinations. In a similar manner we found the $[\Delta^2]$ for the other pairs, and then taking their sum obtained

$$\Sigma\,[\Delta^2] = 11{\cdot}086,\ p = 27,\ \text{and}\ [n] = 157.$$

(5 determinations were lost).

Substituting in [75], we find the probable error of observation for any one determination to be

$$e = \pm \sqrt{\frac{0{\cdot}455\,(11{\cdot}086)}{157 - 27}} = \pm\,0{\cdot}197$$

The probable error of the declinations of the two stars 2330 and 2369 were $\epsilon_{\delta_1} = \pm\sqrt{\cdot 04}$ and $\epsilon_{\delta_2} = \pm\sqrt{\cdot 12}$ respectively, hence the weight assigned to this pair in combining it with the others was

$$w = \frac{n}{n\,(\epsilon^2_{\delta_1} + \epsilon^2_{\delta_2}) + 4\,e^2} = \frac{6}{6\,(\cdot 04 + \cdot 12) + 4\,(0{\cdot}197)^2} = 5{\cdot}41$$

The following is a summary of the results for a few of the pairs, showing a convenient arrangement for the computation of the final latitude and probable error:

No. of Pair	No. of Obs.	ϕ	$[\Delta^2]$	w	$w\phi$	v	v^2	wv^2
1	6	37° 47′ 48·47″	0·243	8·7	73·689	+·17	·029	0·252
2	5	49·22	0·176	5·7	49·554	+·92	·846	4·822
*	*	*	*	*	*	*	*	*
25	5	48·56	0·066	4·8	41·088	+·26	·068	0·326
26	6	48·03	0·152	5·4	43·362	−·27	·073	0·394
27	6	48·72	0·209	5·1	44·472	+·42	·176	0·898
	157	37° 47′ 48·30″ weighted mean	11·086 $\Sigma[\Delta^2]$	179·4 $[w]$	1489·022 $[w\phi]$			25·640 $[wv^2]$

Substituting $[w\ v^2]$ and $[w]$ in equation [79], we have

$$\epsilon_{\phi_0} = \pm\ 0{\cdot}050''$$

AZIMUTH.

725. Definitions. The *azimuth* of an object is the angle which the plane of the meridian makes with the vertical plane which passes through the object whose direction is desired. It is usually reckoned from the south point through the west, north, and east.

An *astronomical azimuth* is the angle obtained by observation upon some celestial body at any one point, and is consequently affected by the local deflections of the plumb line or vertical.

A *geodetic azimuth* is the mean of several astronomical azimuths which were observed at a number of points and connected by a system of triangulation. In a geodetic azimuth it is supposed that the several local deflections will have neutralized each other.

Azimuths are designated primary, secondary, or subordinate, according to the purpose for which they are intended. When used for primary or secondary triangulation they should be obtained with the same accuracy with which the directions or angles are determined. When used for subordinate purposes an accuracy of $\pm\ 1'$ may be sufficient in some cases, as when observing the magnetic declination.

The local time must be known very closely for the best class of work, and is usually determined with a transit while making the azimuth observations for primary triangulation. For other classes of work it is not quite so necessary to have an accurate determination of the time if a close circumpolar star is observed. For subordinate azimuths, when only the nearest minute is desired, the

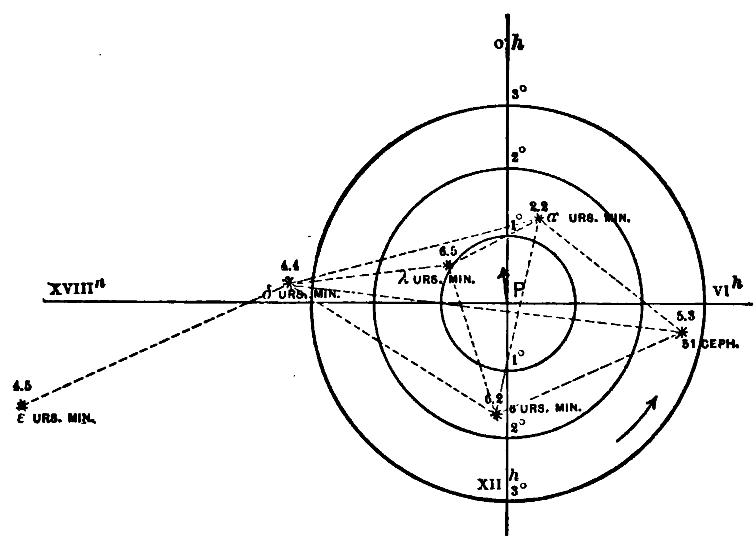
sun's limbs may be observed and the time obtained from these same observations; or it may be obtained from nearly all telegraph offices, as it is now the practice to send a signal over nearly all telegraph lines at mean noon of the standard meridian of the locality in which the observatory sending the signals is situated.

The longitude of the place where the observations are made must be known, however, in order to use this telegraph time.

When an astronomical azimuth is desired the theodolite used on the triangulation is usually preferred, observations being taken in the same manner as the directions or angles, so as to get conditions as nearly like those under which these directions or angles are observed as possible.

A transit or meridian telescope fitted with an eyepiece micrometer may be used in place of the theodolite, but extreme care is necessary to have them both exactly centered over the same point. The azimuth mark should never be much within a mile for the larger

FIG. 472.



telescopes, so that it can be seen distinctly with the sidereal focus—i. e., focus for stars or distant objects.

In practice, the determination of an astronomical azimuth is the measurement of the horizontal angle between a terrestrial object,

usually called the azimuth mark, and some known celestial body, noting the instant of the observation by a chronometer whose error on local time must be determined. The horizontal angle between the vertical plane of the body observed and the plane of the meridian for the instant of the observation may be found from special formulas and the data given in ephemerides; hence we obtain the angle between the azimuth mark and the meridian—i. e., its azimuth—by combining the observed angle with this computed one.

When a primary azimuth is desired, one of the close circumpolar stars is observed, usually either α, δ, or λ Ursæ Minoris, or 51 Cephei. α Ursæ Minoris is large enough to be visible during daylight for telescopes having a magnifying power greater than about 50 or 60 diameters, provided the atmosphere is fairly clear and steady. It is consequently used more than any other.

The accompanying diagram, Fig. 472, shows the positions of these stars with reference to the pole, and also their magnitudes. The direction and length of the small arrow indicates the apparent precessional motion of the pole in one hundred years.

Best Time to Make the Observations. When the time is accurately known the celestial body should be observed near the meridian, provided the altitude is small. When the meridian passage is at a great altitude the horizontal axis of the telescope must be very accurately leveled, and its inclination at the time of the observation obtained and corrected for, if good results are desired. When the time is not known, best results may be obtained by observing a close circumpolar star at both elongations. In general, effects of small errors in declination and latitude are eliminated by taking the mean of results equally distant and on opposite sides of the meridian. To eliminate the effects of small errors in time and right ascension, combine observations at upper and lower culminations.

When the azimuth is obtained from an observed altitude the observations should be made as near the prime vertical passage as possible, provided it is not too high (near the zenith), nor so low as to be affected by changes of refraction. The refraction for altitudes less than 10° is very uncertain, and where fair results are desired the object should be at least 15° or 20° above the horizon.

726. Azimuth by an Observed Altitude of a Star or the Sun. The formulas for determining the azimuth are given in Art. 668. The one where A is found by its tangent is the most accurate, and is consequently used more than any other.

The following example illustrates the usual method of observation and computation:

Station, Capital, Washington, D. C. Sun near prime vertical, August 15, A. M., 1856. Observer, C. A. S. Instrument, 5-inch theodolite. Longitude, $5^h\ 8^m\ 1^s$ west of Greenwich. Thermometer, 73°. Barometer, 30 inches. Vertical circle reads zenith distances. Horizontal circle readings increase from right to left, opposite to motion of hands of clock.

CHRONOMETER TIME.*	HORIZONTAL CIRCLE.		VERTICAL CIRCLE.	
	A	B	A	B
	☉'s upper and first limb. Telescope D.			
$5^h\ 2^m\ 53{\cdot}0^s$	25° 24′ 30″	205° 24′ 30″	61° 56′ 00″	61° 56′ 00″
5 34·0	25 50 45	205 51 30	61 24 30	61 25 00
6 55·5	26 04 30	206 05 15	61 08 45	61 09 30
	☉'s lower and second limb. Telescope R.			
$5^h\ 9^m\ 12{\cdot}0^s$	205° 54′ 15″	25° 54′ 00″	61° 19′ 30″	61° 18′ 30″
10 32·0	206 07 15	26 06 45	61 04 00	61 03 00
11 42·0	206 18 30	26 18 15	60 50 00	60 49 45

Sun's horizontal parallax $\chi = 8{\cdot}5''$. Mean chronometer time * $= 5^h\ 7^m\ 48{\cdot}1^s$.

$$
\begin{array}{lrrrl}
\text{Vertical circle} = & 61^\circ & 17' & 02'' & \\
\text{Refraction} = r = & +1 & 41{\cdot}7 & & \\
\text{Parallax} = & & -7{\cdot}4 & & \\
z = \text{corr. zen. dis.} = & 61^\circ & 18' & 36 & \\
\phi = & 38 & 53 & 18 & \\
\delta = & 13 & 55 & 33 & \\
 & 2)114 & 07 & 27 & \\
s = & 57 & 03 & 43{\cdot}5 & \cos. = 9{\cdot}735383 \\
(s-\delta) & 43 & 08 & 10{\cdot}5 & \sin. = 9{\cdot}834888 \\
(s \sim z) = & 4 & 14 & 52{\cdot}5 & \sec. = 0{\cdot}001194
\end{array}
$$

* A sidereal chronometer was used. The time is only required for taking δ from the ephemeris, and need not be given exact. When a star is used no record of the time is required.

$(s \sim \phi) =$	18 10 25·5	cos. =	0·505985
	$\tan.^2 \frac{1}{2} A =$		2)0.077450
	$\tan. \frac{1}{2} A =$		0·038725
	$\frac{1}{2} A = 47° 33' 04''$		
Sun E. of N. = A =	95 06 08		
Horizontal circle reads	25 56 40		
Circle reading for N. meridian	290 50 32		
Circle reading for S. meridian	110 50 32		

If we had taken a reading upon some terrestrial signal either before or after pointing on the sun (or star), or preferably both before and after, we find the azimuth of this signal by taking the difference between the mean of the readings upon it and the above reading for the meridian.

If greater accuracy is desired the star might be observed direct and also reflected from an artificial horizon, thus eliminating error due to inclination of the axis of the telescope.

727. Azimuth by Observations on Close Circumpolar Star at any Hour Angle. This is the method in most general use. The method of observation is very little different from that in which the angles or directions are observed for the triangulation, the principal difference being to note the instant of observing the star by a clock, and also determining the inclination of the horizontal axis of the telescope by means of a striding level, unless the star is observed reflected as well as direct.

General formula is

$$\tan. A_n = \frac{\sin. t}{\cos. \phi \tan. \delta - \sin. \phi \cos. t} = \frac{\tan. p \sec. \phi \sin. t}{1 - \tan. p \tan. \phi \cos. t}$$

Example. Astronomical azimuth for primary triangulation. Instrument, 20-inch theodolite. $\phi = 39° 06' 51{\cdot}0''$, $\lambda = 2^h 17{\cdot}6^m$ W. of Washington. Observer, W. E. Record is same as on page 58, with also record of the time of observing the star and readings of the striding level before and after.

As the telescope of this instrument is fitted with two parallel vertical cross-hairs, a pointing and reading upon the star is made with each and the mean used as the reading upon the star. When the star is near elongation two independent pointings and readings

are usually made, one on each thread; but when the star is near culmination it moves so rapidly that it is best to note the time of transit over each thread, reading the microscope micrometers but once for both. The mean of the two times, then, is the time of observing the star and corresponds with the circle reading.

The following computation is made, using the last form of the above equation, as it is usually more simple when Albrecht's "Tables" can be had, where we can find log. $\frac{1}{1-a}$ for the argument log. a;

$$\text{The level correction, } C = (\Sigma w - \Sigma e)\frac{d}{2n}\tan. h$$

n is the number of levels $(w - e)$ read;

h is the star's altitude at instant of observation;

d is the value of one division of the level in seconds of arc.

For Polaris, h may be found for this purpose from

$$h = \phi \pm 75' \cos. t, \pm \text{ for} \begin{cases} \text{upper culmination.} \\ \text{lower} \quad \text{"} \end{cases}$$

The last term will be given in minutes of arc.

The following is the computation of one series, or a single determination of the azimuth. An independent pointing and reading was taken for each thread. The direct and reversed readings are computed separately, and the combination of these results is the result for this series. Each of the other series is computed in a similar manner, and the final results combined for a mean value. If the various series have different weights, the combination is similar to that for combining results for ϕ, explained on page 207. The probable error may be obtained as explained there also. It requires very careful work to obtain an azimuth with a probable error less than 0·15″ by observing 30 to 40 series in at least half this number of positions.

SOLAR DATE, POSITION AND SERIES.	Direct or reversed.	Chronometer time of observation.	Mean of chronometer times and chronometer correction.	Sidereal time of observation and star's R. A.	Hour angle in time and arc.	Log. sin. t, and log. tan. p sec. ϕ.	Log. cos. t and log. tan. p tan. ϕ.	Value of $p=90°-\delta$ and logs.
1890 Position I....	D	$15^h\ 21^m\ 11·25^s$	$15^h\ 25^m\ 08·75^s$	$15^h\ 25^m\ 10·81^s$	$14^h\ 06^m\ 08·94^s$	9·718288n	9·93070n	1° 16′ 50·37″
Series 1.....		29 06·25	ΔT=+02·06	1 18 66·87	211° 30′ 59·1″	(8·459583)	(8·25952)	log. tan. p=8·349383
	R	38 22·25	$15^h\ 40^m\ 49·75^s$	15 40 51·83	$14^h\ 21^m\ 44·95^s$	9·763287n	9·91103n	log. tan. ϕ=9·910138
Aug. 20·22...		43 17·25	ΔT=+02·08	1 18 66·88	215° 26′ 14·3″			log. sec. ϕ=0·110200

Culmination.	LOGS.		Log. tan. A_n and A_n.	Circle reading upon star.	Mean circle reading and level correction.	Circle reading corrected, and A_n.	CIRCLE READINGS.	D or R.	AZIMUTH OF MARK.	
	Cos. t tan. ϕ tan. p =log. a.	Sin. t, sec. ϕ, tan. p. and log $(1-a)$ (Table)					True N. and azimuth mark.		W. of N.	
		8·177871n	8·171191n	76° 10′ 41·50″	76° 12′ 07·15″	76° 12′ 11·44″	75° 21′ 12·41″	D	75° 21′ 19·01″	75° 21′
L. C.	8·19022n	−0·006680	−50′ 59·03″	13 32·80	$+24·1^d=+04·29$	−50 59·08	359 59 53·40			18·99″
		8·222870n	8·216485n	16 55·10	76° 17′ 45·60″	17 42·11	75 21 06·87	R	18·97″	
	8·17055n	−0·006385	−56′ 35·24″	18 36·09	$-19·6^d=-08·49$	−56 35·24	359 59 47·90			

728. Diurnal Aberration. When results are desired for primary work a correction must be applied for diurnal aberration. This may be obtained from

$$k = \frac{0{\cdot}308'' \cos. A \cos. \phi}{\sin. z}$$

For elongation, $k = 0{\cdot}308'' \cos. A$ with sufficient accuracy. This correction is always positive to an azimuth reckoned in the usual way, from left to right, or from south to west, etc.

729. Observation with a Transit Instrument of a Close Circumpolar Star near Culmination. 1. When the eyepiece is fitted with a micrometer which can be used with the movable thread vertical. An azimuth mark must be set as near the meridian as possible. (It should rarely be farther from the meridian than about one quarter of the visible field of the telescope.) The angle between this mark and the meridian is then measured with the micrometer screw. The meridian reading is obtained by noting the times of successive transits of a close circumpolar star over the micrometer wire, which is moved ahead of the star a known number of divisions each time as soon as the star passes it. The chronometer correction Δ T must be known very accurately, but by combining observations made above and below the pole any small errors in Δ T or α may be practically eliminated.

In order to determine or eliminate the instrumental errors and diminish the effect of the errors of observation, the following method of observation is usually preferred: Level and adjust instrument carefully, then make a number of independent pointings upon the mark with the micrometer wire, noting the reading of the micrometer in each case. It is best to begin as late as possible, so that the instrument has not time to change before observing the star. Twenty minutes before the star crosses the meridian is usually long enough. Next take a reading of the striding level in both positions, then observe the transits of the star, moving the micrometer wire 50 divisions each time, say, but only far enough so as to give time to get good deliberate transits. The distance the micrometer wire must be set ahead each time will vary with the distance of the star from the pole. At present λ Ursæ Minoris is the nearest to

the pole of the well-determined stars, and hence the micrometer intervals may be the smallest when observing this star. Before beginning the observations the reading of the line of collimation should be obtained roughly by taking the mean of direct and reversed readings on the mark, then when the star approaches this reading reverse the telescope and note the transits over the wire placed successively in the same positions as before reversal, only in reverse order. By this means the error of collimation is eliminated. As soon as the transits of the star are completed the level should be read, and then another set of readings taken on the mark. If it is desirable to take all the transits on the star without reversing, the correction for collimation may be found from the direct and reversed readings on the mark.

Reduction.—Let T represent the mean of all the times of transit for the star, and m the mean of the corresponding micrometer readings; $B\,b =$ correction for level obtained same as on page 170, and also including the pivot inequality.

$m_1 =$ mean of all the readings on the mark for telescope D.

$m_2 =$ " " " " R.

$m_o = \frac{m_1 + m_2}{2} =$ reading of line of collimation.

$(m_o - m_1)\,R$, or $(m_2 - m_o)\,R =$ angle between mark and line of collimation, R being the value of one division of the micrometer.

R may be found from the observations on the star by determining the time t required by the star to cross one division, whence

$$R = 15\,t \cos. \delta$$

The correction for collimation, when the telescope is not reversed in the middle of the observations on the star, may be obtained by taking the difference between m and m_o and converting it into time, using the value for t, as found when getting the value of R; or, if we have a very large correction, and wish to use the mean value found for R_o from a number of observations, we have the collimation correction c from

$$c = \frac{(m_o - m)\,R_o}{15 \cos. \delta}$$

We can now find the true sidereal time T_o of the observation from

$$T_o = T + \Delta T + B\,b + c$$

c is zero, of course, when the telescope is reversed in the middle of the observations on the star, the micrometer readings being the same after reversal as before.

The hour angle t of the star at the instant of crossing the line of collimation is found from

$$t = a - T_0$$

We now compute the azimuth of the star at this instant from the fundamental equation [1], substituting for z its equivalent $(\delta - \phi)$, since the star is practically in the meridian, whence

$$A_n = \frac{\cos.\,\delta \operatorname{cosec.} (\delta - \phi) \sin.\, t}{\sin 1''}$$

For lower culmination we must use $180° - \delta$ in place of δ.

EXAMPLE OF RECORD.

Station, Depot Key, Fla. March 20, 1852. δ Ursæ Min. at L. C., and 51 Cephei at U. C. Observer J. E. H. Instrument, the Simms Transit, C. S. No. 8. Chronometer correction, $\Delta T = -51{\cdot}80^s$.

Time by chron. 202, sid.	Mark. Microm. Lamp E.	Mark. Microm. Lamp W.	Time by chron. 202, sid.	δ Ursæ Min. Chron. time by 202, sid.	δ Ursæ Min. Microm.	51 Cephei. Chron. time by 202, sid.	51 Cephei. Microm.	Mark not observed after observing the stars.
				Lamp W.	Lamp W.	Lamp W.	Lamp W.	
h. m.	*t. d.*	*t. d.*	*h. m.*	*h. m. s.*	*t. d.*	*h. m. s.*	*t. d.*	
5 20	18 76·0	12 67·0	5 40	6 18 44	18 22	6 27 34	13 22	
	76·0	66·5		19 11	17 72	28 08	13 72	
	75·0	67·0		19 37·5	17 22	28 40·5	14 22	
	76·0	66·5		20 04·5	16 72	29 15	14 72	
	75·0	67·5		20 31·5	16 22	29 48	15 22	
	75·0	67·5		20 59	15 72	30 22	15 72	
	75·1	67·0		21 25·5	15 22	30 54	16 22	
	75·1	67·2		21 52	14 72	31 29	16 72	
	75·8	67·0		22 19	14 22	32 01·5	17 22	
5 30	75·0	66·5	5 50	22 46	13 72	32 36	17 72	
				23 13	13 22	33 10	18 22	
Means.	18 75·4	12 67·0						

Level for δ Ursæ Min. W.	Level for δ Ursæ Min. E.		Level for 51 Cephei. W.	Level for 51 Cephei. E.
48·1	49·8	$\phi = 29°\ 07'\ 30''$	63·0	38·0
63·0	35·0	1 div. of level = 1″	49·0	53·5
48·2	50·8	100 divs. of microm. = 1 turn.	63·5	39·0
63·0	36·3		49·0	53·5

Reduction.—Determination of the value of one division of the micrometer.

Mean of times, $6^h\,20^m\,58{\cdot}5^s$ for δ Ursæ Minoris, and $6^h\,30^m\,21{\cdot}6^s$ for 51 Cephei.

Corresponding micrometer readings, $15^t\,72^d$ for δ Ursæ Minoris, and $15^t\,72^d$ for 51 Cephei; hence the following differences from the mean:

δ URSÆ MIN.			51 CEPHEI.		
m. s.	t. d.	From obs. of δ Urs.	m. s.	t. d.	From obs. of 51
2 14·5	2 50	Min.:	2 47·6	2 50	Cephei:
1 47·5	2 00	1 div. corresponds	2 13·6	2 00	1 div. corresponds
1 21·0	1 50	to 0·5383s	1 41·1	1 50	to 0·6703s
0 54·0	1 00	log. 15.. 1·17609	1 06·6	1 00	log. 15.. 1·17609
0 27·0	0 50	log. t.... 9·73102	0 33·6	0 50	log. t.... 9·82627
0 00·5	0 00	log. cos. δ 8·77395	0 00·4	0 00	log. cos. δ 8·67961
0 27·0	0 50		0 32·4	0 50	
0 53·5	1 00	9·68106	1 07·4	1 00	9·68197
1 20·5	1 50		1 39·9	1 50	
1 47·5	2 00	1 div.=0·4798″	2 14·4	2 00	1 div.=0·4808″
2 14·5	2 50		2 48·4	2 50	
Sum 807·5s	1500 div.		1005·4s	1500 div.	

Mean of all measures: 1 *division of micrometer* = 0·4800″, as obtained from several other sets as well as these.

$$
\begin{aligned}
m_1 &= 12^t\,67{\cdot}0^d \qquad & m_2 - m_o &= +3^t\,04{\cdot}2^d \\
m_2 &= 18\ \ 75{\cdot}4 & (m_2 - m_o)\,R &= +2'\,26{\cdot}02'' = \\
m_o &= 15\ \ 71{\cdot}2 & &\text{Angle between line of collima-} \\
m &= 15\ \ 72{\cdot}0 & &\text{tion and mark.} \\
m_o - m &= \ \ -0{\cdot}8
\end{aligned}
$$

$$\text{For } \delta \text{ Ursæ Minoris, } c = \frac{-0{\cdot}8\,(0{\cdot}48'')}{15 \cos.\,(93^\circ\,24'\,24'')} = +0{\cdot}42''$$

$$\text{For 51 Cephei, } \qquad c = \frac{-0{\cdot}8\,(0{\cdot}48'')}{15 \cos.\,(87^\circ\,15'\,33'')} = -0{\cdot}54''$$

$B = -7{\cdot}30$ δ Ursæ Minoris. $\quad B = +11{\cdot}04$ 51 Cephei.

	h m s		h m s
Chronometer time T	= 6 20 58·46		6 30 21·64
ΔT	= −51·30		−51·30
Level cor. = 0·42s × 7·30 =	−03·07	0·337 × 11·04 =	× 03·72
Collimation cor. c	= +00·42		−00·54
T_o	= 6 20 04·51		6 29 33·52
a	= 6 20 05·61		6 29 33·15
$a - T_o = t =$	+ 1·10s = +16·50″		− 0·37s = −5·55″

51

sin. t	$= 5{\cdot}90286$	$5{\cdot}42987$
cos. δ	$= 8{\cdot}77395^n$	$8{\cdot}67961$
cosec. $(\delta - \phi)$	$= 0{\cdot}04530$	$0{\cdot}07095$
sin. A_n	$= 4{\cdot}72211^n$	$4{\cdot}18043$
A_n	$= -1{\cdot}09''$	$-0{\cdot}31''$

which is the azimuth of the line of collimation (or the star as it crossed the line of collimation) from the north. The minus sign indicates that it is west of north. Above, we find the angle between line of collimation and mark to be $+ 2' 26{\cdot}02''$; hence azimuth of the mark from the north $= 2' 24{\cdot}93''$ and $2' 25{\cdot}71''$. The correction for diurnal aberration is still to be applied.

2. In case the eyepiece of the telescope has no micrometer attachment, the azimuth may still be determined by this same method if there is a micrometer attachment to one of the wyes. This pivot micrometer, however, can not be moved very far without tending to elevate the axis in the wyes, and is therefore not very convenient to use.

If desirable, the azimuth may be determined by observing a close circumpolar star near elongation, measuring a micrometric angle between the star and a mark placed near the vertical plane of the elongation of the star. It is not used very much, however, as only one star can be observed usually with the same mark, and this at only one elongation. It is also more difficult to set the mark in the correct position, as most transit instruments are not provided with any means for measuring horizontal angles. The method is very fully described in "Coast and Geodetic Survey Report" for 1891, Appendix II. It may also be found in some treatises on astronomy.

CHAPTER XIII.

TRIGONOMETRIC LEVELING.

730. Introduction. Trigonometric leveling is the determination of elevations by means of vertical angles and horizontal distances.

The trigonometric method of determining heights is of great advantage in a hilly or mountainous country where triangulation is in progress, as but little extra work is necessary to make the vertical angle observations while observing the horizontal angles; and if observations are obtained on as many as ten or fifteen days, and the lines are not too long, the results are perhaps as good as could be obtained by "precise spirit leveling" in such country. If the country is flat, however, with but slight elevations, precise spirit leveling is unquestionably the more accurate, since the refraction on long lines near the surface of the earth is too variable and uncertain for the determination of accurate results by trigonometric leveling. Even in mountain regions, where the lines are elevated, the refraction is very unsteady, and changes much more rapidly on some parts of the line than on others; hence better results may be expected from short lines than from long ones.

731. Instruments. Two forms of instruments are used for the measurement of vertical angles: the vertical circle, and the ocular micrometer of the telescope of a large theodolite. The latter is used to measure the small vertical angle between two stations, or several stations, which are very nearly in the horizon of the instrument, or at about the same distance vertically from this horizon, so that these small differential angles are within the range of the micrometer screw. (For the description of the micrometer, see page 44.) The telescope of the instrument is supposed to make the same angle with the vertical in each position, or else a level is

attached to it so as to determine the changes in its inclination which may occur between the pointings, and corrections applied to the resulting angles for such changes.

The angle thus observed is in terms of turns and divisions of the micrometer screw, hence, in order to obtain it in arc, the value of the screw in seconds of arc must be obtained. This is done either by measuring an angle accurately determined by means of the horizontal circle or other angle measurer, or by the astronomical method explained in Art. 719.

Some form of gradienter could be used in a similar manner to the ocular micrometer if it were large enough to give accurate

FIG. 473.

results, and would be of more general advantage, since greater differences could be measured with it than with the ocular micrometer.

For a method of combining micrometric angles with zenith distances, see page 237.

732. Vertical Circle. The vertical circles in use at present differ but slightly from that shown in Fig. 473. In principle their construction is similar to that of a theodolite, the circle being used in a vertical plane in place of the horizontal, as on the theodolite.

There are two types of vertical circles, the repeating and non-repeating. The repeating circle is usually preferred, as it enables the observer to multiply the measures of an angle before reading the circle, thus lessening the effects of errors of graduation and reading, and also gives a result which is a mean of any changes in refraction that may be taking place during the lengthened period of observation, which is always preferable, since it is impossible to obtain the actual effect of the refraction, or entirely eliminate it.

The circle, in either type of instrument, is usually placed on the side of the vertical axis with a counterpoise on the opposite side. The telescope is rigidly fastened to the vernier plate, or part that corresponds to the alidade of the engineer's transit. A sensitive level is fixed on the vertical axis just behind the circle, so as to determine the perpendicularity of the vertical axis.

733. Method of Observation. The instrument should be carefully leveled, so that the bubble will remain in practically the same place for all positions around the horizon; then point on the object whose angle is to be observed, having the telescope on the right, and read the circle and level; next revolve the instrument horizontally 180°, bringing the telescope on the left; unclamp the vernier plate, revolve the telescope vertically, and again point on the object, using the slow-motion screw belonging to the vernier plate. This makes one measure of the double zenith distance of the object. Next, revolve back to the first position, using the clamp and tangent screw of the outer circle for the first pointing, and then proceed as before. The usual practice on the Coast and Geodetic Survey is

to make three such repetitions,* and then take another reading of the circle and level. The difference of the two circle readings divided by 6 and corrected for level gives the angle formed by the intersection of the normal to the earth's surface at the instrument and the line of sight from the instrument to the object observed, or the zenith distance, as it is called.

Another such set of three repetitions of the double zenith distance taken immediately following will serve as a check upon the first, and the two, if they do not differ more than $3''$ or $4''$, are usually sufficient to obtain the angle for that day if taken between 10 A. M. and 3 P. M.

734. Level Correction. As it is practically impossible to preserve the verticality of the axis of the instrument while making the observations, it is customary to read the level at the beginning and end of each set and correct the resulting zenith distance for the inclination thus developed. This correction (c), to be applied to the zenith distance resulting from a set of observations, may be obtained from the following formula,

$$c = \frac{b}{4}[(e + e') - (o + o')] \qquad [1.]$$

where o and e represent the readings of the ends of the level bubble at the object and eye ends of the telescope respectively at the beginning of the set, o' and e' the corresponding readings at the end, and b the value of one division of the level in seconds of arc. This formula applies only when the level divisions are numbered from the center toward the ends. In case a level is used with the divisions numbered consecutively from one end toward the other, as from left to right, then the formula becomes

$$c = \frac{b}{4}[(e' - e) - (o - o')] \qquad [2.]$$

735. Reduction to Center. In actual field work the vertical circle, for various reasons, is often mounted eccentrically, and it

* It is usually preferable to make six repetitions of an angle, if any are made, as the computation is simplified thereby, it being merely necessary to write down the remainder, after dividing each denomination of the angle, as the first figure of the next denomination.

becomes necessary to reduce each observed zenith distance to what it would be if the instrument were at the station mark and the object observed at its station mark.

This reduction to the station mark may be considered as made up of three separate corrections:

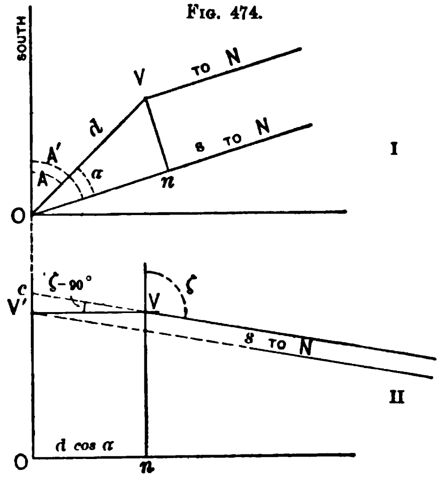
1. The correction due to curvature, or difference of zeniths of the vertical-circle station and the triangulation station.

2. That due to the inclination of the line of sight to the horizontal.

3. The correction due to the difference in height between instrument and station mark, and also between the object observed and its station mark.

As the eccentricity of the vertical-circle station is usually very small compared with the distances of the stations observed, the corrections for the reduction to center of the zenith distances are small, and it is only necessary to use approximate methods to obtain them.

Fig. 474.



In Fig. 474 I, let O represent the center of the station, V the vertical-circle station, N a station observed from V, A and A′ the azimuths of the lines to the vertical circle and N respectively, $\alpha = A' - A$, d, the distance of V from O, and s the distance from O to N.

First Correction. Since V and O are two different points on the surface of the earth, their verticals are not parallel, but form an angle at the center of the earth equal to that subtended by the line d. If the station observed is in the prolongation of the line O V, its observed zenith distance must be corrected by the angle at the center of the earth subtended by d, in order to reduce it to the station O. Since s is very large compared with d, V N and O N

may be considered parallel, then O n will subtend the angle necessary to correct the observed zenith distance of N from V in order to reduce it to O. O $n = d \cos. a$.

An angle, c, at the center of the earth subtended by a small arc is equal to $\frac{\text{arc (in linear measure)}}{\rho \sin. 1''}$, ρ representing the radius of curvature of the earth at this point. Hence the required correction

$$c_1 = \frac{\text{O}\, n}{\rho \sin. 1''} = \frac{d \cos. a}{\rho \sin. 1''}$$

Assuming $\rho = 6{,}370{,}000$ metres, which is about the average value in the United States, we have, nearly enough for our purpose,

$$c_1 = \frac{d \cos. a}{30} \qquad [3.]$$

Second Correction. In Fig. 474 II let the points and lines bearing the same letters as in Fig. 474 I represent the relative positions of those objects when projected on the vertical plane of O N, the horizons of O and V being considered the same, as the curvature will not affect the correction we are deducing.

Let ζ represent the observed zenith distance of N at V, O V′ = n V the elevation of the vertical circle above the station mark, and C the intersection of the line of sight, V N, with the perpendicular at O. Then it is evident that to transfer ζ from V to V′ it must be corrected for a quantity equivalent to the assumption of a change of V′ C in the elevation of either V or N.

Now, V′ C = O n tan. $(\zeta - 90°) = d \cos. a \tan. (\zeta - 90°)$, hence the correction to ζ in seconds of arc is

$$c_2 = \frac{d \cos. a \tan. (\zeta - 90°)}{s \sin. 1''} \qquad [4.]$$

Third Correction. Let H = height of object observed at N above the station mark, and V the height of the vertical-circle telescope above the station mark O; then H − V is the effective height above the station mark of the object observed at N, for which a correction must be applied to the observed zenith distance of N from O in order to reduce it to the station marks at both stations. Since this height is ordinarily very small, we may obtain the correction in seconds of arc from

$$c_3 = \frac{\text{H} - \text{V}}{s \sin. 1''} \qquad [5.]$$

Total Correction. Combining equations [3] [4] and [5], we have for the total correction, c, in seconds of arc, necessary to reduce to the station marks a zenith distance observed eccentrically, and also with the instrument and object observed upon, both at elevations differing from the elevation of the station marks,

$$c = c_1 + c_2 + c_3 = \frac{d \cos. \alpha}{30} + \frac{d \cos. \alpha \tan. (\zeta - 90°) + \mathrm{H} - \mathrm{V}}{s \sin. 1''} \qquad [6.]$$

the unit of length being the metre.

The signs of the angular functions must be strictly attended to, as also the signs of H and V, the two latter being negative when the instrument and object observed are below the respective station marks.

If the natural tangent for $(\zeta - 90°)$ be used, and a table giving seconds of arc for the arguments, difference of height in centimetres (numerator of last term in the equation), and s in kilometres, the computation is facilitated considerably.

In order that the first correction may amount to 0·1″, the effective eccentricity ($d \cos. \alpha$) of the vertical circle must be 3 metres; hence it may be neglected excepting in refined work, or where the eccentricity is large.

The second correction is always small, and may be neglected excepting in the most refined work, or where the observed zenith distance differs much from 90°, with a large eccentricity of the vertical circle.

736. Refraction. The ray of light joining two stations is always more or less curved, owing to refraction, while passing through the air. As the densities of the air and the amount of moisture suspended therein are constantly changing, the refraction of the ray of light joining two stations must also be variable. Numerous observations made on a number of days have shown that the refraction of a line of sight is greatest during the night, becoming a maximum a little before daybreak, and then rapidly diminishing until about 9.30 or 10 A. M. From 10 A. M. to about 3 P. M. the change is usually slight. The minimum, on an average, however, occurs about 1 P. M. At about 3 P. M. the refraction begins to increase, and changes rapidly until about an hour after dark, then changes

slowly for the remainder of the night. This change is perhaps more plainly indicated by the curves in Fig. 475, which represent the diurnal variation in the coefficient of refraction in several localities. Observations were made hourly for a number of days (more than ten, with few exceptions). The value for each hour as plotted is the mean of all the observations made at that hour during the days of observation.

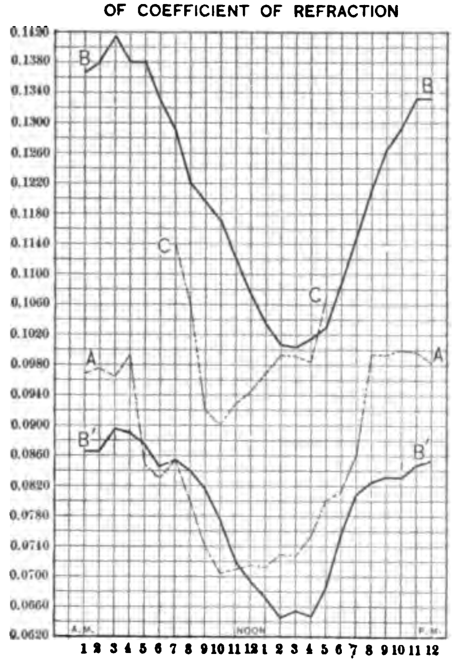


Fig. 475.

CURVES SHOWING DIURNAL VARIATION OF COEFFICIENT OF REFRACTION

Curve A is the result of observations made at Ragged Mountain, Me., during July, August, and September.

Curve B, the result of observations at Martinez East, near Straits of Carquines, California.

Curve B′, at Mount Diablo, California.

Curve C, the mean of the results at Bodega Head and Ross Mountain, on the coast of California, about fifty miles north of San Francisco.

Owing to the excessive moisture on the Pacific coast the refraction is greater and more variable than east of the Sierra Nevada Mountains, as is distinctly shown by these curves.

B and B′ were obtained from observations at opposite ends of the same line, B being at an elevation of about 57 metres, and B′ about 1,173 metres above the sea level. B and B′ show that the refraction is much larger and more variable at low elevations than at high, and that the changes are in practically the same direction.

When to Observe.—Experience has shown that the best results for double zenith distances are obtained by making the observations

at about noon or 1 P. M.—i. e., during the period of minimum refraction. Two sets, as mentioned above, are usually sufficient to get the zenith distance on any one day. As the refraction is very different on different days, observations must be made on a number of days in order to get the zenith distance for an average refraction. In mountain work, with long lines, it has been found that ten or fifteen days' observations are necessary, and that it is not advisable to use observations made on particularly stormy days, or immediately after a heavy storm, as the refraction is usually very abnormal at such times.

Determination of Refraction.—In Fig. 476, let

s = sea-level distance between two stations, A and B, O being the center of the earth, and C, C′ the mean sea-level surface.

h_1 and h_2 = heights of A and B respectively, above surface, C, C′;

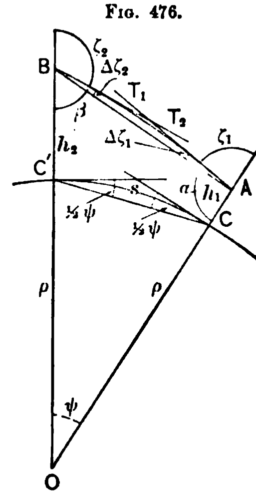
ρ = radius of curvature of earth's surface for the line s;

ψ = angle at earth's center subtended by the line s;

ζ_1 and ζ_2 = observed zenith distances at A and B, upon B and A respectively;

$\Delta\zeta_1$ and $\Delta\zeta_2$ = corresponding refraction angles.

FIG. 476.



Let the curved line A B represent the line of sight, whatever the nature of the curve may be. Ordinarily it differs but little from the arc of a circle.

A T and B T are tangents to this curve at A and B respectively,

$$\alpha = 180° - (\zeta_1 + \Delta\zeta_1)$$
$$\beta = 180° - (\zeta_2 + \Delta\zeta_2)$$

Since the curve A B differs but little from the arc of a circle, we can make

$$\Delta\zeta_1 = m_1\psi, \qquad \Delta\zeta_2 = m_2\psi,$$

where m_1 and m_2 are called *coefficients of refraction.* Hence the coefficient of refraction at either end of a line joining two stations is the ratio of the refraction angle at that end to the angle at the center of the earth subtended by the arc separating the two stations.

If a number of such results are obtained in any locality, and the weighted mean be taken, the result is considered the coefficient of refraction for that locality.

To find the Coefficient of Refraction.—1. From the geometrical relations of Fig. 476 we have

$$\zeta_1 + \Delta\zeta_1 + \zeta_2 + \Delta\zeta_2 = 180° + \psi \qquad [8.]$$

Substituting for $\Delta\zeta_1$ and $\Delta\zeta_2$ their values as given above, [8] becomes

$$m_1 + m_2 = 1 - \frac{1}{\psi}(\zeta_1 + \zeta_2 - 180°) \qquad [9.]$$

As ψ is always a small angle, we may substitute $\frac{s}{\rho \sin. 1''}$ for it without appreciable error, whence

$$m_1 + m_2 = 1 - \frac{\rho \sin. 1''}{s}(\zeta_1 + \zeta_2 - 180°) \qquad [10.]$$

If the observations at the two stations are simultaneous, and the two stations differ but little in elevation, we may assume $m_1 = m_2 = m$, and obtain the mean refraction coefficient for this line,

$$m = 0{\cdot}5 - \frac{\rho \sin. 1''}{2s}(\zeta_1 + \zeta_2 - 180°) \qquad [11.]$$

2. When we have a net of lines of trigonometric leveling with reciprocal observations, all those at any one station being nearly simultaneous, it is much better to assume that the value of m at each station of the net is a constant, since the refraction varies with the atmospheric pressure (and hence with the elevation), and also with differing hygrometrio conditions; and it is much more probable that these conditions are more uniform on all the lines radiating from any one station than at the opposite ends of any one line. Then form all equations similar to [10] that are possible in the net, and solve by the method of least squares for m_1, m_2, m_3, ... m_n.

If the observations on all the lines at any one station are not nearly simultaneous, or if the character of the country on one side of the station is different from that on the other, it may be necessary to use two or more values for m at the same station. The number of equations, however, will nearly always surpass the number of unknowns, so as to admit of a solution by least squares.

In assigning weights to the equations we can take the form used on the United States Coast and Geodetic Survey, it being equivalent to that given by Wright in his work on "Adjustment of Observations." If ζ_1 were observed n_1 times, and ζ_2 n_2 times, the weights of ζ_1 and ζ_2 may be taken to be n_1 and n_2 respectively; then each of the equations of the form of [10] would have a weight p where

$$\frac{1}{p} = \frac{\rho^2 \sin.^2 1''}{4 s^2} \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \qquad [12.]$$

—i. e., its weight would be proportional to $\frac{n_1 n_2}{n_1 + n_2} s^2$.

3. When the elevations of two stations are known, by precise leveling or other means, we can determine the value of m at either end of the line joining the two stations, at any time, with an accuracy dependent principally upon the errors of observation.

Let z_1 and z_2 be the true zenith distances at A and B upon B and A respectively, then we have, from Fig. 476,

$$z_2 - z_1 = (\zeta_2 + \Delta \zeta_2) - (\zeta_1 + \Delta \zeta_1) = (\zeta_2 - \zeta_1) + \frac{m_2 - m_1}{\rho \sin. 1''} s$$

The last member of this equation will be found in equation [17], which is used for the determination of the differences of height. Since h_1 and h_2 are known, we can transpose equation [17], introducing $z_2 - z_1$ for its equivalent, as given in the above equation, and we have

$$\tfrac{1}{2}(z_2 - z_1) = \tan.^{-1} \left\{ \frac{h_2 - h_1}{s \left(1 + \frac{h_2 + h_1}{2\rho} + \frac{s^2}{12\rho^2} \right)} \right\}$$

But $\frac{h_2 + h_1}{2\rho}$ and $\frac{s^2}{12\rho^2}$ are always very small; hence we may write the equation

$$\tfrac{1}{2}(z_2 - z_1) = \tan.^{-1} \left\{ \frac{h_2 - h_1}{s} \left(1 - \frac{h_2 + h_1}{2\rho} - \frac{s^2}{12\rho^2} \right) \right\} \qquad [13.]$$

From Fig. 476 we see that

$$\tfrac{1}{2}(z_2 + z_1) = 90^\circ + \tfrac{1}{2}\psi = 90^\circ + \frac{s}{2\rho \sin. 1''} \qquad [14.]$$

From these two equations we can compute the true zenith distances at either end of the line; hence we can obtain the coefficient

of refraction at either station at any instant by using the observed zenith distance at that instant in the following equations:

$$z_1 - \zeta_1 = \Delta\,\zeta_1 = m_1\,\psi = \frac{m_1\,s}{\rho\,\sin.\,1''}$$

$$z_2 - \zeta_2 = \Delta\,\zeta_2 = m_2\,\psi = \frac{m_2\,s}{\rho\,\sin.\,1''}$$

whence

$$m_1 = \frac{(z_1 - \zeta_1)\,\rho\,\sin.\,1''}{s} \qquad [15.]$$

$$m_2 = \frac{(z_2 - \zeta_2)\,\rho\,\sin.\,1''}{s} \qquad [16.]$$

This method was used in the determination of m at Martinez East and Mount Diablo, the results of which are shown in the curves B and B′ on page 228. For a full discussion of the method employed, see Appendix 12, "United States Coast and Geodetic Survey Report" for 1883.

In combining the various values of m for a mean value we may assign relative weights as described in the second method, or use the empirical formula proposed by Bessel and often used by the Coast and Geodetic Survey, viz.,

$$\frac{n_1\,n_2}{n_1 + n_2}\sqrt{s}$$

Numerical Values of Coefficient of Refraction.—The coefficient of refraction differs considerably in different localities, being greatest over water or along the coast, and least in the interior and at high elevations.

The following are a few of the values obtained for the minimum refraction in the immediate locality named. Each result is the mean of the values obtained at several stations, observations being made at each for a number of days.

LOCALITY.	OBSERVED BY	MEAN m.
Maine	U. S. C. and G. S.	0·0710
North Georgia	U. S. C. and G. S.	0·0715
Central California	U. S. C. and G. S.	0·0742
New York State	New York State Survey.	0·0730

On the Coast and Geodetic Survey the following values are used for field computations, or in any work where refinement is unnecessary.

Over parts of the sea near the coast,	$m = 0{\cdot}078$
Between primary stations,	$m = 0{\cdot}071$
In the interior of the country,	$m = 0{\cdot}065$

These values, it may be said, are too small to be used west of the Sierra Nevada Mountains, as shown by the curves on page 228.

The term primary station is used in this place, as it is always understood, on the Coast and Geodetic Survey, that the stations are elevated so that the lines of sight pass above the region of excessive atmospheric disturbance near the surface of the ground.

737. Derivation of Formulas for Computation. There are three cases to be considered:

1. When the zenith distances are observed at both ends of each line (i. e., are reciprocal) but are not simultaneous.

2. When the zenith distances are reciprocal and *also* simultaneous.

3. When observations of zenith distances are made at only one end of a line.

First Case.—From the triangle A O B, Fig. 476, we have

$$\frac{\text{O A} - \text{O B}}{\text{O A} + \text{O B}} = \frac{\tan. \frac{1}{2} (\beta - \alpha)}{\tan. \frac{1}{2} (\beta + \alpha)}$$

But $\text{O A} = \rho + h_1$, $\text{O B} = \rho + h_2$, $\alpha = 180° - (\zeta_1 + \Delta \zeta_1)$, $\beta = 180° - (\zeta_2 + \Delta \zeta_2)$, and $\alpha + \beta = 180° - \psi$. Substituting these values in the above equation, we get

$$\frac{h_1 - h_2}{h_1 + h_2 + 2\rho} = \tan. \tfrac{1}{2} \psi \tan. \tfrac{1}{2} \left[(\zeta_1 - \zeta_2) + (\Delta \zeta_1 - \Delta \zeta_2) \right] \quad [16a.]$$

Substituting for $\Delta \zeta_1$ and $\Delta \zeta_2$ their values as given on page 229, developing the term $\tan. \frac{1}{2} \psi$ by Maclaurin's series and reducing, we get

$$h_1 - h_2 = s \tan. \tfrac{1}{2} \left[(\zeta_1 - \zeta_2) + \frac{m_1 - m_2}{\rho \sin. 1''} s \right] \left(1 + \frac{h_1 + h_2}{2\rho} + \frac{s^2}{12\rho^2} + \ldots \right) \quad [17]$$

Second Case.—When the zenith-distance observations are reciprocal and also simultaneous, m_1 and m_2 are usually considered equal, particularly in mountain regions where the stations are elevated and not far from the same height. Equation [17] then becomes

$$h_1 - h_2 = s \tan. \tfrac{1}{2} (\zeta_1 - \zeta_2) \left(1 + \frac{h_1 + h_2}{2\rho} + \frac{s^2}{12\rho^2} + \text{etc.} \right) \quad [18.]$$

which is the form used most generally on long lines. Even on the longest lines, however, the effect of the second factor of the equation is rarely larger than two metres, and the last term of this factor may usually be neglected without appreciable error, since the error of assuming $m_1 = m_2$ is undoubtedly greater.

The factor $\left(1 + \frac{h_1 + h_2}{2\rho} + \frac{s^2}{12\rho^2} + \text{etc.}\right)$ may be tabulated for the arguments $h_1 + h_2$ and s, so its effect on a certain difference of elevation as found by the first part of the formula can be seen by inspection and applied very quickly.

Another formula used occasionally is

$$h_1 - h_2 = \frac{s \sin. \frac{1}{2}(\zeta_1 - \zeta_2)}{\cos. \frac{1}{2}(\zeta_1 - \zeta_2 + \psi)} \qquad [19.]$$

obtained directly from the geometrical conditions of the figure.

Third Case.—When the zenith-distance observations are made at one end of the line only, or it is desirable to determine the difference in elevation of two stations from each station independently of the other.

From Fig. 476 we have

$$\zeta_2 + \Delta\zeta_2 = 180° + \psi - (\zeta_1 + \Delta\zeta_1)$$

and by substituting this value for $\zeta_2 + \Delta\zeta_2$ in equation [16a] and transposing, we get

$$h_1 - h_2 = \left(1 + \frac{h_1 + h_2}{2\rho}\right) 2\rho \tan. \tfrac{1}{2}\psi \cot. (\zeta_1 + \Delta\zeta_1 - \tfrac{1}{2}\psi) \qquad [20.]$$

Since ψ is always a small angle, we may put $s = 2\rho \tan. \frac{1}{2}\psi$, and substituting it, together with $\Delta\zeta_1 = m_1 \frac{s}{\rho}$ and $\frac{1}{2}\psi = \frac{s}{2\rho}$, in [20].

$$h_1 - h_2 = s\left(1 + \frac{h_1 + h_2}{2\rho}\right) \cot. \left(\zeta_1 - \frac{(1 - 2m_1)s}{2\rho}\right) \qquad [21.]$$

Developing $\cot. \left(\zeta_1 - \frac{(1 - 2m_1)s}{2\rho}\right)$ by Taylor's theorem, and putting $\sin.^2 \zeta_1 = 1$, since ζ_1 never differs much from 90°, we get

$$h_1 - h_2 = s\left(1 + \frac{h_1 + h_2}{2\rho}\right)$$
$$\left[\cot. \zeta_1 + \frac{1 - 2m_1}{2\rho} s + \left(\frac{1 - 2m_1}{2\rho}\right)^2 s^2 \cot. \zeta_1 + \ldots\right] \qquad [22.]$$

The factors $\frac{1 - 2m_1}{2\rho}$ and $1 + \frac{h_1 + h_2}{2\rho}$ may be tabulated and the equations solved quite rapidly.

Other forms of equations have been used in this case also, notably

$$h_1 - h_2 = s \cot. \zeta_1 + \frac{1 - 2m_1}{2\rho} s^2 + \frac{1 - m_1}{\rho} s^2 \cot.^2 \zeta_1 + \text{etc.} \qquad [23.]$$

obtained by substituting

$$\tan. \left(a + \frac{1 - 2m}{2\rho} s\right) \text{for} \cot. \left(\zeta_1 - \frac{1 - 2m}{2\rho} s\right)$$

in the above development by Taylor's theorem;

also $$h_1 - h_2 = \frac{s \cos. (\zeta_1 + m\psi - \frac{1}{2}\psi)}{\sin. (\zeta_1 + m\psi - \psi)} \qquad [24.]$$

The tabulated values of $\log. \frac{1}{2}(1 - 2m)$ and $\log. (1 - m)$ for values of m varying from 0·050 to 0·100 may be found in Appendix 18, "Coast and Geodetic Survey Report" for 1876.

738. Leveling by the Sea Horizon. It occasionally becomes desirable to obtain the elevation at a point by observing the sea horizon, hence we must deduce a formula applicable to the case.

In triangle A C C′, Fig. 477, using the same notation as in Fig. 476,

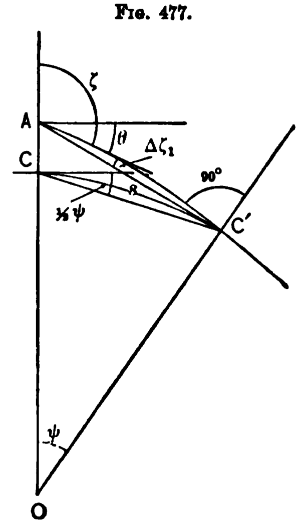
$$h \sin. (90° + \tfrac{1}{2}\psi) = \text{A C}' \sin. \tfrac{1}{2}\psi.$$

Since $\text{A C}' = \rho \tan \psi$,

we have

$$h = \rho \tan. \psi \frac{\sin. \frac{1}{2}\psi}{\cos. \frac{1}{2}\psi} = \rho \tan. \psi \tan. \tfrac{1}{2}\psi = 2\rho \frac{\tan.^2 \frac{1}{2}\psi}{1 - \tan.^2 \frac{1}{2}\psi} \qquad [25.]$$

Fig. 477.



Since ψ is very small, we may put

$$\tan.^2 \tfrac{1}{2}\psi = \frac{\psi^2}{4} \sin.^2 1''$$

without appreciable error; whence [25] becomes

$$h = \tfrac{1}{2}\rho\psi^2 \sin.^2 1'' \left(\frac{1}{1 - \frac{1}{4}\psi^2 \sin.^2 1''}\right) \qquad [26.]$$

Developing $\dfrac{1}{1 - \frac{1}{4}\psi^2 \sin.^2 1''}$ by the binomial formula, we get

$$h = \tfrac{1}{2}\rho\psi^2 \sin.^2 1'' (1 + \tfrac{1}{4}\psi^2 \sin.^2 1'' + \tfrac{1}{16}\psi^4 \sin.^4 1'' + \text{etc.}) \qquad [27.]$$

From the geometrical conditions of the figure, we see $\psi =$

$\theta + \Delta \zeta_1$, θ being the angular dip of the sea horizon, and we have, from page 229, $\Delta \zeta_1 = m \psi$;

hence $$\psi = \theta + m\psi = \frac{\theta}{1 - m}$$

which, substituted in [27], dropping all terms involving higher powers of ψ than the second, gives

$$h = \tfrac{1}{2} \rho \frac{\theta^2}{(1 - m)^2} \sin.^2 1'' \left(1 + \tfrac{1}{4} \frac{\theta^2}{(1 - m)^2} \sin.^2 1''\right) \qquad [28.]$$

Approximations.—If only approximate elevations are desired, or if the lines are not more than ten miles in length, we can use the following formula in place of equation [22], without greater error than our uncertainty, ordinarily, in the coefficient of refraction:

$$h_1 - h_2 = s \cot. \zeta_2 + k s^2$$

where $$\log. k = 2{\cdot}3128 \,(-10) \text{ in feet,}$$
$$= 2{\cdot}8288 \,(-10) \text{ in metres,}$$

obtained by using $m = 0{\cdot}07$, and the average value of ρ for latitude 45°, viz.,

$$\log. \rho = 6{\cdot}80470 \text{ in metres,}$$
$$= 7{\cdot}32068 \text{ in feet.}$$

Dip and Distance of Sea Horizon.—In equation [28], assuming

$$m = 0{\cdot}0784 \text{ and } \log. \rho = 6{\cdot}80470 \text{ in metres,}$$
$$= 7{\cdot}32068 \text{ in feet,}$$

we get $$\theta \text{ (in sec. of arc)} = 58{\cdot}82 \sqrt{h} \text{ in feet,}$$
$$= 106{\cdot}54 \sqrt{h} \text{ in metres,}$$
$$s \text{ (in miles)} = 1{\cdot}317 \sqrt{h} \text{ in feet,}$$
$$s \text{ (in kilometres)} = 3{\cdot}839 \sqrt{h} \text{ in metres.}$$

739. Radius of Curvature. Since the radius of the earth's curvature is needed in nearly all geodetic work, the following table, based upon the Clark spheroid of 1866, is inserted. It gives the value of log. ρ for each two degrees of latitude from 24° to 50°, and for each five degrees of azimuth. If a more extended table is desired one may be found in the "Coast and Geodetic Survey Report" for 1876, page 386. The last figure of the numbers in these tables is slightly uncertain, owing to our lack of knowledge as to the exact shape of the earth.

TABLE OF LOGARITHMS OF RADIUS OF THE EARTH'S CURVATURE.

	Azimuth.	Latitude. 24°	26°	28°	30°	32°	34°	36°
	°							
Meridian......	0	6·802479	6·802597	6·802722	6·802852	6·802988	6·803129	6·803274
	5	2498	2615	2789	2869	3004	3145	3289
	10	2553	2669	2791	2919	3052	3190	3332
	15	2644	2756	2875	3000	3130	3265	3404
	20	2766	2875	2990	3111	3236	3366	3500
	30	3093	3192	3296	3405	3518	3636	3757
	40	3496	3580	3671	3766	3864	3967	4072
	50	3923	3994	4070	4150	4233	4319	4407
	60	4325	4384	4446	4512	4580	4650	4723
	70	4653	4702	4753	4807	4863	4921	4980
	75	4776	4822	4869	4918	4969	5022	5076
	80	4867	4909	4953	4999	5047	5097	5148
	85	4923	4963	5006	5049	5096	5143	5192
Perpendicular..	90	6·804942	6·804981	6·805023	6·805066	6·805112	6·805159	6·805207

		38°	40°	42°	44°	46°	48°	50°
Meridian......	0	6·803422	6·803573	6·803726	6·803880	6·804035	6·804189	6·804342
	5	3436	3586	3739	3892	4045	4199	4351
	10	3478	3626	3775	3926	4077	4228	4378
	15	3546	3690	3835	3982	4130	4277	4423
	20	3637	3776	3917	4059	4201	4343	4484
	30	3880	4006	4133	4262	4391	4519	4647
	40	4179	4289	4400	4511	4623	4735	4846
	50	4498	4590	4683	4777	4871	4965	5058
	60	4797	4873	4949	5025	5104	5181	5257
	70	5041	5104	5166	5229	5293	5357	5420
	75	5133	5190	5248	5307	5364	5423	5481
	80	5201	5254	5308	5363	5417	5472	5526
	85	5242	5294	5345	5397	5450	5502	5554
Perpendicular..	90	6·805256	6·805307	6·805358	6·805409	6·805460	6·805512	6·805563

Combination of Micrometric Angles with Zenith Distances.—When micrometric angles are observed they must be referred to one or more of the stations, and hence are merely the relative angles between those stations, thus giving no means for the determination of the absolute elevations; hence zenith distances must be observed on at least one of the stations at each station. As but little more work is necessary to observe two or three stations than one after the instrument is in position, it often happens that we have both micrometric angles and zenith distances observed on the same stations, and it becomes necessary to combine them for the most probable values.

The following example illustrates the method used on the Coast and Geodetic Survey, and is a simple application of least squares

for adjusting the small differences, or, in other words, for combining the two sets of values, each having its own weight.

The weights, p, are usually assigned relative to the absolute measures and in proportion to the number of observations.

Let A, B, and C be three stations upon which both zenith distances and micrometric angles have been observed from station D.

The micrometric angles are all referred to A as zero.

Let $v_1, v_2, \ldots$ to v_6 represent the required corrections for each of the observed quantities respectively.

Stations.	OBSERVED VALUES.						ADJUSTED VALUES.	
	Zenith dis.	Corr'n.	Wt.	Mic. Ang.	Corr'n.	Wt.	Zenith dis.	Mic. angles.	
	° ′ ″		p			p	° ′ ″	″	″
A	90 50 10	$+v_1$	2	00	$+v_4$	4	90 50 10·4	− 0·2	0·0
B	49 30	$+v_2$	2	−35	$+v_5$	4	49 33·7	−36·9	−36·7
C	51 20	$+v_3$	1	+59	$+v_6$	4	51 11·7	+61·1	+61·3

$$\text{Condition equations, } 0 = +\ \ 5 + v_1 - v_2 - v_4 + v_5$$
$$0 = +11 - v_1 + v_3 + v_4 - v_6$$

CORRELATE EQUATIONS.

	$\frac{4}{p}$	C_1	C_2	
v_1	2	+1	−1	$v_1 = +0{\cdot}4''$
v_2	2	−1		$v_2 = +3{\cdot}7$
v_3	4		+1	$v_3 = -8{\cdot}3$
v_4	1	−1	+1	$v_4 = -0{\cdot}2$
v_5	1	+1		$v_5 = -1{\cdot}9$
v_6	1		−1	$v_6 = +2{\cdot}1$

NORMAL EQUATIONS.

	C_1	C_2
$0 = +5$	+6	−3
$0 = +11$	−3	+8

whence $C_1 = -1{\cdot}87$ and $C_2 = -2{\cdot}08$

The condition equations are formed by equating the differences between stations A and B and A and C, as obtained in column of zenith distance + correction and column of micrometric angles + correction. For the explanation of the formation of the correlate equations, the normal equations, methods of solving the latter, and, finally, the determination of $v_1, v_2, \ldots$ to v_6, the student is referred to any good work on least squares, and particularly to Wright's "Adjustment of Observations," Chapter V.

740. Precision of Trigonometric Leveling. The precision of the differences of height resulting from trigonometric leveling is usually obtained by the application of the method of independ-

ently observed quantities as developed by Wright in his "Adjustment of Observations," page 105 and following.

If equations [17], [18], and [22] are differentiated, taking s, ζ_1 and ζ_2, m_1 and m_2 as independent variables, considering ζ_1 and ζ_2 each 90° (which they are very nearly), and that we may put $d\,s = 0$, since distances are well known in comparison with heights, we have

$$\mu^2_{h_1} = \tfrac{1}{4}\, s^2 \sin.^2 1''\, \mu^2_\zeta + \tfrac{1}{2} \frac{s^4}{\rho^2} \mu^2_m \qquad [29.]$$

$$\mu^2_{h_2} = \tfrac{1}{2}\, s^2 \sin.^2 1''\, \mu^2_\zeta \qquad [30.]$$

$$\mu^2_{h_3} = s^2 \sin.^2 1''\, \mu^2_\zeta + \frac{s^4}{\rho^2} \mu^2_m \qquad [31.]$$

from the law of errors, where μ_{h_1}, μ_{h_2} and μ_{h_3} are the "mean errors" (called mean square errors by Wright) of the difference in height of two stations from nonsimultaneous reciprocal observations [29], from simultaneous reciprocal observations [30], and from observations at one station only [31]. μ_ζ and μ_m are the "mean errors" of the observed zenith distances and coefficients of refraction. μ_ζ may be obtained from the observed values of the zenith distances by the formula

$$\mu^2_\zeta = \frac{[v\,v]}{n\,(n-1)}$$

μ_ζ being the "mean error" of the arithmetic mean of the observed values, n in number, and $[v\,v]$ the sum of the squares of all the residuals obtained by taking the difference between each observed value and the mean of all. If the observations are of unequal weight we must use the formula

$$\mu^2_{\zeta_w} = \frac{[p\,v\,v]}{[p]\,n\,(n-1)}$$

where μ_{ζ_w} is the "mean error" of the weighted mean, and $[p]$ the sum of the individual weights, p.

μ_m may be determined by differentiating the formula used in deriving m and treating it in a manner similar to the derivation of equations [29], [30], and [31].

The equations [29], [30], and [31] show that the best results may be expected from simultaneously and reciprocally observed zenith distances, which is what we would naturally infer since the

lines of sight from either end of a line are more likely to be identical then than under any other conditions.

The probable error (ϵ) of a result is about two thirds of the mean error, or, more accurately, $\epsilon = \pm 0{\cdot}6745\,\mu$.

Adjustment of a Net of Trigonometric Levels.—When the best possible results are desired from reciprocally observed zenith distances, and nonsimultaneous, we may use the method given in Wright's "Adjustment of Observations," pages 388–391.

Such refinement, however, is rarely used, the observations usually being treated as simultaneous whenever they are reciprocal.

After computing the differences of height over all the lines of the net by either equation [18] or [22] (preferably the former), small discrepancies will be developed when we determine the elevation of a station by using different routes. In order to remove the possibility of such a *contretemps*, all the possible discrepancies of the net are developed and then distributed among the various lines by an application of the method of least squares. The development of the discrepancies is usually by the formation of the condition equations, which result from the condition that in every closed figure the sum of the differences of height must equal zero—i. e., if we start from any station and proceed through several lines, finally returning to the first station, its elevation as computed through these lines should be the same as that started with. The equation for each figure is formed in a manner similar to the following: Suppose the differences of elevation of three stations have all been computed, and are $+126$, -160, $+30$, proceeding consecutively from A to B, from B to C, and from C to A; then, if v_1, v_2, v_3 are the corrections required to make the sum of these three quantities equal zero, we have the equation of condition

$$0 = +126 + v_1 - 160 + v_2 + 30 + v_3$$

or

$$0 = -4 + v_1 + v_2 + v_3$$

Then forming all such equations possible in the figure, or net of lines, taking care to always use the same relative sign for the correction of the difference of height as that used for the difference of height itself (i. e., both must change signs if either does), we can solve them for v_1, v_2, etc., by the methods of least squares. Applying these corrections to the computed differences of height, we

obtain the adjusted values, which may be combined in any way desired without showing discrepancies.

Weights.—The weights, p, to be assigned to the above differences of height can be obtained from the formula

$$\frac{1}{p} = \mu^2{}_h = \tfrac{1}{2}\, s^2 \sin.^2 1'' \mu^2{}_\zeta$$

for simultaneous reciprocal observations, and for observations at one end of the line only we have

$$\frac{1}{p} = \mu^2{}_h = \tfrac{1}{2}\, s^2 \sin.^2 1'' \quad \mu^2{}_\zeta + \tfrac{1}{4} \frac{s^4}{\rho^2} \mu^2{}_m$$

Hence with simultaneous reciprocal observations we may weight inversely as the square of the distance.

With observations in one direction only we must weight inversely as the square of the distance for distances up to about four miles, and inversely as the fourth power for greater distances, since by using average values for $\mu^2{}_\zeta$ and $\mu^2{}_m$ in the last equation we find that the first term is the more important for distances up to four miles, and the second term for greater distances.

For more information on this subject the student is referred to Wright's "Treatise on the Adjustment of Observations," Jordan's "Handbuch der Vermessungskunde," Helmert, "Hoeher Geodäsie," also "Coast and Geodetic Survey Reports" for 1876 and 1884.

It sometimes happens that it is necessary to publish a list of heights with a close degree of approximation while the observations are still incomplete. In this case the approximate method of adjustment employed on the Ordnance Survey of Great Britain may be employed. See also Helmert, "Ausgleichungsrechnung," page 154 *et seq.*, and "Report of New York State Survey," 1882.

For reduction to orthometric heights and correction for inclination of the plumb line, see chapter on Precise Spirit Leveling.

CHAPTER XIV.

PRECISE SPIRIT LEVELING.

741. Definition. Precise spirit leveling is the determination of the relative heights of points on the earth's surface with the utmost degree of accuracy by means of specially devised leveling instruments and rods.

All elevations must be referred to some fixed or datum surface, which is taken by nearly all nations as the mean level of the surface of the ocean; hence, in order to determine accurately the elevations of inland points some distance from the coast, we must have some means of carrying a level line a long distance without the accumulation of errors.

Accurate elevations are needed for a great many purposes, particularly for the reduction of geodetic operations to the reference surface, mean sea level; to aid, by comparing tidal planes along the coast, in the solution of important questions relating to the ocean; to form the basis and to bring into accord all topographic surveys; to furnish data required in gravity work; and to supply points of reference for engineering operations of all kinds.

The degree of accuracy desirable for a few of the more apparent objects, as given in the report of the Geodetic Conference ("Coast and Geodetic Survey Report," 1893, page 305), is:

For topography, within 1 metre.

For base-line reduction, within 0·5 metre.

For gravity operations, within 0·6 metre.

For meteorological investigations, within 1 foot, or 0·3 metre.

For physical hydrography and tidal planes the utmost degree of accuracy attainable.

For the study of the strata and flow of water (underground) in arid regions, and all engineering operations, great accuracy is desirable.

In the United States leveling of a precise character has been done by the Mississippi and Missouri River Commissions along those rivers, and also from the Mississippi to Chicago, and from New York to Duluth; by the United States engineers in connecting the Great Lakes, and in connection with river improvements; and various lines by the United States Coast and Geodetic Survey, namely, from Sandy Hook to Salina, Kan. (with a branch line from Hagerstown, Md., to Old Point Comfort, Va.), two lines from this line south to the Gulf of Mexico, a line from St. Augustine to Cedar Keys across the neck of the peninsula of Florida, and several other short lines.

742. Instruments. The precise or geodetic level, as now in use in various countries, differs from the ordinary wye or engineer's level in several respects. The principal differences, however, are that the precise level is more carefully constructed, and usually provided with a fine, smoothly cut screw for moving the telescope vertically about a horizontal axis, and the level so arranged that it may be easily reversed to eliminate errors of adjustment.

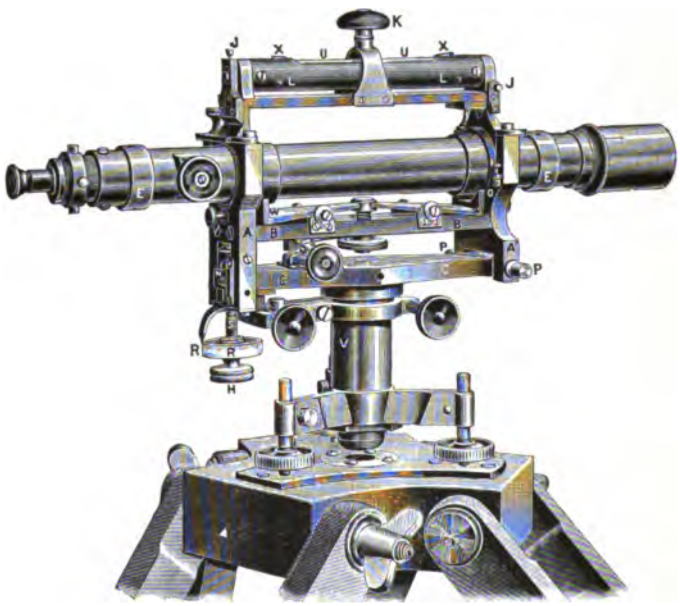
Among the precise levels, as among the wye levels, numerous forms are in existence, each having its advocates.

The Kern level has been used very extensively in Europe, and also in this country on the United States Lake Survey and Mississippi and Missouri River Surveys, and by the United States engineers. It was the instrument prescribed by the International Geodetic Commission held in Berlin in 1864.

The geodetic level, as now used on the Coast and Geodetic Survey (Fig. 478), consists of two uprights, A A', terminating in wyes at the top for support of the telescope, T, and rigidly fastened together by a horizontal bar, B B, just below the telescope. Immediately below this bar is another, C C, which is rigidly fastened in the center to the vertical axis of the instrument, V, and at the object end to the upright, A', by an adjustable pivot, P P, on each side. At the eye end it has a nut, N (between movable guide pieces at the lower end of the upright, A), through which a micrometer screw, S, works. This screw abuts, through a small cap, against the top of the opening between the guide pieces of the upright,

thus enabling the observer to raise or lower that end of the superstructure at will. The micrometer screw is made of steel, and has forty turns to the centimetre. It is provided with a zylonite reading head, *R*, divided into one hundred equal parts, and has a milled rubber head, *H*, underneath for use in turning. Increasing read-

FIG. 478.



ings on the micrometer head indicate an elevation of the eye end of the telescope. There is a cam hook, *M*, between the two plates which enables the observer to raise the upper part of the superstructure off the micrometer screw, and which is kept in position by means of a spring, *S*, while transporting the instrument. Two false wyes, *W W*, attached to the upper plate enable one to raise the telescope from its wyes also during transportation. The collars of the telescope, where resting upon the wyes, are made of hard bell metal, and consequently wear very little. Two adjustable screws, *O*, attached to the wye at the object end, and a pin, *I*, in the side of the telescope, define two positions of the telescope in which the observing wire is horizontal when turning the telescope about its optical axis.

The reticule in the telescope has four horizontal spider lines and one vertical, the observing wire being in the center, with two wires on one side and one on the other for use as stadia wires.

Zylonite bands, *E E*, are placed near the ends of the telescope so that the observer can revolve the telescope in its wyes without touching the tube itself, thus avoiding the effect of temperature changes due to this cause. A detached striding level, *L L*, resting on the collars of the telescope, is used. It has a hard-rubber handle, *K*, for use in reversing, is adjustable by means of abutting screws, *J J*, and is so balanced as to bear with equal weight on the telescope collars. The level vial, *U U*, is mounted in a tube and rests upon two pins at each end, and is held down by a third at each end, being pressed against the top by means of a small spring, *X X*, thus allowing the vial to move longitudinally, for any changes in temperature, without distorting or changing its curvature. All the sensitive levels now in use on the Coast and Geodetic Survey are mounted in this manner.

The tripod is of wood, with a head considerably larger than the foot of the instrument to secure greater stability. The other parts may be seen in the illustration.

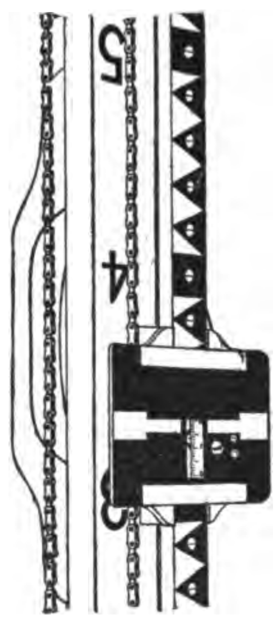
The instrument, including tripod, weighs about 11 kilogrammes (23 pounds). Focal length of telescope, 24·8 centimetres. Aperture, 2·9 centimetres. Magnifying power, 28. Value of one division of the level, 2 millimetres = about 2″.

743. Rods. Previous to 1895 the rods used by the Coast and Geodetic Survey consisted of a thin metal scale about three metres long, and supported by being let into a wooden rod the cross-section of which was a cross. Pointings were made on a target carrying a small scale which moved over the face of the metal scale. The metal scale was fastened rigidly only at the bottom of the wooden part of the rod, so as to allow free expansion upward for temperature changes.

In 1894 an entirely new rod was devised by the Survey and used very satisfactorily during the season of 1895. Fig. 479 shows the target and a short section of the rod, and Fig. 480 shows the lower end of the rod. It is made of three strips of white pine wood,

thoroughly seasoned, fastened together in the form of a cross with symmetrical proportions, and is a little more than three metres long. Brass plugs 2 centimetres long are sunk in the face of one of the projecting pieces of the rod at intervals of 2 centimetres, and the graduation marks for 2 centimetres intervals of the rod placed upon them, their surfaces being silvered for that purpose. These plugs fit accurately in the wood, and are held in place by a rivet through the wood and rear end of the plug. They project slightly above the wood, so that a millimetre scale 2 centimetres long in the center of the target can have its feather edge pressed against the face of the plugs, and thus enable the observer to read without parallax. A spring keeps this small scale away from the face of the plugs when not in use. The zero of this scale corresponds with the center of the target. The target is provided with guide pieces and friction springs, and is moved by means of an endless chain working over a pulley at each end of the rod, the upper being adjustable for varying lengths of chain. The target is clamped, without taking the rod down or loss of time, by means of an endless chain attached to a lever and eccentric carried by the target and passing over a pulley at each end of the rod. The temperature is registered by a thermometer let into the wood of the rod. A circular level attached to the back of the rod enables the rodman to keep it in a vertical position. The face of the rod is divided, by painting black and white, into centimetre divisions to serve for telemeter readings, and also to check the rodman's reading of the rod.

FIG. 479.

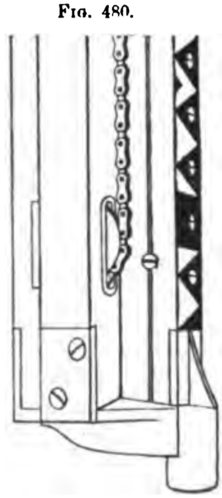


The bottom of the rod is made of metal, and terminates in a rounded boss of phosphor bronze with a radius of 2·7 centimetres. It is so placed that the point of support is in the same vertical plane as the graduation.

The most important feature of this rod is its preparation for protection against the action of moisture. Hitherto the principal objection to wood for measuring rods has been its action under varying conditions of moisture. Although painted, oiled, varnished, or otherwise treated, it was never satisfactory. This rod has been saturated with paraffin. After the wooden parts had been cut and nearly finished ready for the metal parts it was immersed in a bath of boiling paraffin and kept there for several hours, then allowed to cool in the paraffin. The heat drove out the resinous parts of the wood in the form of gas, and when allowed to cool the paraffin penetrated to the very center of the wood, as shown on pieces of the same wood which were treated in this manner and then cut up for inspection.

As a test of its hygroscopic properties, after removal from the paraffin bath the rod was carefully compared as to length and thickness with a standard, and then submerged in a trough of water for nineteen hours and then again compared, but no appreciable difference was developed.

FIG. 480.



As the expansion of wood for changes of temperature is so much smaller than that of metal, it is always preferable when the temperature changes are so large and rapid as often happens on leveling work. Wood is not solid or hard enough, however, to receive fine graduation marks and hold them for any length of time, but this is overcome by the insertion of metal plugs for the graduation, as in the above-described rod. The hygroscopic properties of this rod are as perfect as can be desired, hence the two greatest objections to wood for measuring rods have been overcome, and the rod used by the Coast and Geodetic Survey combines the best properties of both the wooden and metal rods and eliminates the worst of each.

Foot Plates.—The foot plate used with the above-described rod is a circular disk of cast iron, about 15 centimetres in diameter,

with a depression (radius of 3·5 centimetres) in the center for receiving the foot of the rod. Prongs underneath prevent lateral motion when properly pressed into the ground. It is sometimes necessary to use a peg in place of the foot plate in order to obtain a stable support for the rod. The foot of the rod, in this case, rests upon a copper nail driven in the top of the peg.

744. Adjustments. The method of adjusting the level for "wind," and to make it parallel with the plane of its supports, is the same as explained on page 42.

To make the Axis of the Instrument Vertical and Perpendicular to the Axis of the Level.—After adjusting the level, bring the bubble near the center and then revolve the instrument about its vertical axis 180°. If the bubble has moved, bring it back half way by means of the leveling screws, and then bring it to the center by means of the micrometer screw. Repeat until the bubble remains in the center for both positions of the instrument. The reading of the micrometer is then noted, as it is the reading corresponding to a horizontal pointing when the instrument is in adjustment—i. e., when the plane of the upper surfaces of the collars of the telescope is perpendicular to the vertical axis. After once obtaining this reading the micrometer is set to this reading, and the instrument leveled the same as the ordinary wye level.

To make the Middle Thread, or Observing Wire, Horizontal.—Two adjustable abutting screws are attached to the sides of the object wye, and a pin in the side of the telescope is brought into contact with first one, and then, by rotating the telescope, with the other. The object of these abutting screws is to define two positions where the horizontal observing wire is horizontal, or the vertical wire perpendicular. The screws may be raised or lowered until the horizontal wire is horizontal, as shown by noting whether the wire continues to bisect a distant well-defined object when the telescope is moved slightly in azimuth. The vertical thread should then be truly vertical, but it is very difficult to so place the cross hairs on the reticule that the horizontal and vertical threads will be exactly perpendicular to each other, although they are usually so nearly so that it is necessary to compare a considerable part of the

visible portion of the lines in order to detect any error. It is preferable to have the vertical thread perpendicular, so as to detect any inclination in the rod, as errors from this source are much more appreciable than those due to the slight inclination of the horizontal thread, the observations always being made very near the center of the telescope. The abutting screws are then so adjusted that the vertical thread is parallel with a plumb line suspended at a convenient distance, the instrument having been previously leveled. If the abutting screws can not be moved far enough the reticule may be twisted, loosening all its adjusting screws for the purpose.

To adjust the Collimation.—Point the telescope on a convenient mark, clamp rigidly, and then revolve the telescope in its wyes to an inverted position 180° from the first. One half the deviation of the horizontal thread is corrected by means of the micrometer screw, and one half of the deviation of the vertical thread by means of the tangent screw; the remaining deviation of each thread is adjusted by means of the adjusting screws of the reticule. When this adjustment has been made by pointing on a mark at a considerable distance, it should be repeated by pointing on one so much nearer that the focus of the telescope requires changing. If the adjustment remains, the motion of the ocular is parallel to the geometrical and optical axis of the telescope; but if not, a mechanician alone can make the required adjustment.

To determine whether the Telescope moves in a Vertical Plane.—This is an adjustment depending on the construction of the instrument, and may be tested by pointing the telescope on a plumb line and seeing whether the line of sight travels along it when the telescope is raised or depressed by means of the micrometer screw. If it does not, it may be corrected by a readjustment of the pivot screws of the superstructure.

745. Instrumental Constants. In addition to the adjustments just described there are four instrumental constants to be determined.

1. *The Standardization of the Stadia Wires* is the same as described in the discussion of the stadia. A table should be prepared giving the distances corresponding to the spaces intercepted on the rod.

2. *The Angular Value of a Turn of the Micrometer.*—This depends upon the pitch of the screw and the distance between the center of the screw and the pivots about which the superstructure is moved by means of the micrometer screw. The value may be obtained from direct measurement of these quantities, but the best method is to measure the space (d) on a rod or scale set up at a distance (D) from the pivots of the superstructure, traversed by the observing wire when the telescope is raised or depressed by moving the micrometer through a certain number of turns (m). It is usually sufficient to run the micrometer but two turns each side of the horizontal reading, as the screw is rarely, if ever, used at a greater distance in the field. From the similar triangles formed between the two lines of sight we get the value of a turn (t) of the micrometer screw in seconds of arc,

$$t = \frac{d}{m \, \mathrm{D} \sin. 1''}$$

The value of m, corresponding to the space (d) on the rod, should be observed several times in order to eliminate the errors of observation. Having obtained the value of the screw, a table may be prepared giving the space (d) on the rod for the arguments D and m, using the formula

$$d = m \, t \, \mathrm{D} \sin. 1''$$

thus giving the correction necessary to reduce the target reading of the rod to what it should read in the horizon of the telescope.

3. *The Angular Value of One Division of the Level Scale.*—Having obtained the value of the micrometer screw, note the number of divisions (n) that either end of the bubble moves (or preferably the mean of both ends) when the telescope is raised or depressed by moving the micrometer screw through m turns, the level resting upon the collars of the telescope. The value of one division (l) in seconds of arc is given by

$$l = \frac{m \, t}{n}$$

This is the principle of the "level trier" which is used in the determination of nearly all level values.

In the method of observing used by the Coast and Geodetic

Survey, the value of a division of the level is not needed, excepting to show the sensitiveness—i. e., whether the curvature is that which experience has shown to be the best for accurate field work. This is about 1″ to 2½″ to the millimetre, as a more sensitive level is too easily affected by changing atmospheric conditions.

4. *The Inequality of the Collars of the Telescope.*—In order to obtain results at all accordant, the instrument must be mounted very firmly, the telescope so balanced as to bear with equal weight on the wyes, and the level protected from atmospheric changes, heat of the body of the observer, and the like. The instrument is rigidly clamped so that the plane of the wyes is considered immovable. Measure the inclination of the plane of the upper surface of the collars in four different positions, by reading the level direct and reversed while resting upon the collars.

Let I be the inclination in the first position, telescope erect, and object end on the right;

I_1, inclination in the second position, telescope erect, and object end on the left, the telescope having been changed end for end by lifting it from the wyes;

i and i_1 are corresponding inclinations when the telescope is inverted, or revolved 180° about its optical axis.

If o and e, o' and e' represent the readings of the object and eye ends of the level bubble, direct and reversed, respectively;

o_1 and e_1, o_1' and e_1', the readings for the second position, etc.;

Then
$$
\begin{aligned}
I &= \tfrac{1}{4}\left[(o + o') - (e + e')\right] l \\
I_1 &= \tfrac{1}{4}\left[(o_1 + o_1') - (e_1 + e_1')\right] l \\
i &= \tfrac{1}{4}\left[(o_2 + o_2') - (e_2 + e_2')\right] l \\
i_1 &= \tfrac{1}{4}\left[(o_3 + o_3') - (e_3 + e_3')\right] l
\end{aligned}
$$

The mean of all these inclinations, having regard to the signs of the quantities, gives the inclination of the upper surface of the collars with reference to the lower surface, and dividing by 2 we get the inclination with reference to the line of sight of the telescope, a plus sign indicating that the object-end collar is the larger. Hence the inequality of collars (h) is given by

$$h = \tfrac{1}{8}\left(I + I_1 + i + i_1\right)$$

If the object end is the larger (h positive), the line of sight will be inclined downward when the upper surface of the collars is hori-

zontal, and the reading of the rod must be increased. The correction (c) to an observed sight at a distance, D, is given by

$$c = + \mathrm{D}\, h \sin. 1''$$

The collar inequality should be determined at the beginning and end of each season's work, or about once in two months.

746. Standardization and Adjustment of Rods. The rod should be compared with a standard for total length and uniformity of graduation, and its coefficient of expansion determined.

The thermometer should be compared with a standard.

The zero of the millimetre scale should correspond with the center of the target, or, if it does not, a correction must be applied whenever the rod reading is to be referred to a bench mark on which the foot of the rod can not be placed.

The distance of the foot of the rod from the zero point of the graduation, called the index correction, should be determined. It must be applied whenever the rod is used an odd number of times between two bench marks.

The target should be horizontal when the rod is vertical.

The level on the rod should be adjusted to indicate the verticality of the rod. It may be tested by suspending a plummet from the rod.

747. Field Procedure. The various methods of conducting the operations in the field may be classed as follows:

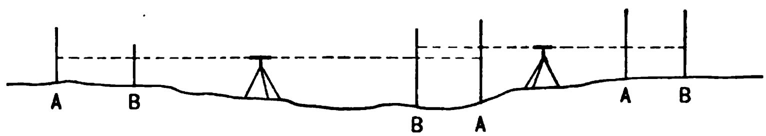
1. A single line, with two rods, in opposite directions by the same observer.
2. A single line, with two rods, in opposite directions with different observers.
3. A simultaneous double line in one direction with one observer.
4. A simultaneous double line, alternate sections in opposite directions by the same observer.
5. A simultaneous double line in one direction, in sections by different observers.

The method now employed by the Coast and Geodetic Survey is No. 3, with a (proposed) similar line in the opposite direction, whenever practicable, as a check.

A simultaneous double line is where two rods are observed in the backsight, and also in the foresight, for each position of the instrument. The successive positions of the instrument and rods are shown in Fig. 481.

The simultaneous double line offers the advantage of a considerable saving of time over a single line in opposite directions, and also affords a check against errors of observation from station to

FIG. 481.



station, since the difference in the readings of the two rods obtained in the backsight should be equal to that obtained in the previous foresight—i. e., the two lines of sight should be parallel excepting for the slight difference due to the earth's curvature and refraction. If the same difference is not obtained then some error has been introduced, either accidental or some constant instrumental error. The latter would show a want of parallelism in the two lines of sight equal to twice its effect in one direction, and should be developed in each position of the instrument and rods.

This method of a simultaneous double line can not be given the same weight, however, as two single lines in opposite directions, since there is but one setting of the instrument for both the determinations of the difference in height between two stations, and any change in the instrument between the backsights and foresights will enter the same in both lines or determinations; also the atmospheric conditions are practically the same for both rods, so that any abnormal state is not developed, thus admitting another source of error.

In case the instrument settles between the backsight and foresight, or the rod between the foresight and backsight, the elevations from there on are all too high by the amount of the settling, hence this source of error is cumulative. Any settling of the rod can usually be detected on the simultaneous double line from the fact that the difference in reading of the two rods should be the same

in the foresight and following backsight. But whenever it is small enough to fall within the error of observation it will be so considered, and therefore may cumulate the whole length of the line. Any errors due to changes in the instrument are probably as effective on the simultaneous double line as on a single line in one direction.

As it is a physical impossibility to read the level and rod at the same instant, another source of error is due to the changes in the level during this interval of time, caused by the proximity of the body of the observer and the constantly changing currents of heated air. This error is likely to be cumulative. Its reduction to a minimum depends upon the skill of the observer and the method of observation.

These cumulative errors are the bugbear of all levelers, and it seems to be impossible to entirely eliminate them.

The mean of two lines in opposite directions is evidently the most simple way to balance these cumulative errors, the supposition being that the cumulation is the same, or nearly so, in each direction.

Personal peculiarities of the observer in handling the instrument, reading the target and level, etc., also enter largely into all leveling work; hence it would seem that a better result would be obtained if a line were leveled by the same observer in both directions rather than by different observers, since differences are observed and not absolute values. In order to get the best possible results, however, and eliminate the effect of personal peculiarities, the line should be leveled in both directions by each of a large number of observers and the mean of all taken.

748. Method of Observation. In order to eliminate the effect of changing refraction and curvature of the earth's surface the instrument is always so placed as to make the foresight and backsight as nearly equal as is practicable. The manner of taking the observation is to bring the target approximately into the horizon of the telescope, and then measure the small vertical angle between the target and the horizon with the micrometer screw by repeated pointings so arranged as to eliminate the level and collimation errors

of adjustment. The target reading of the rod is then reduced to the horizon by applying the correction due to this small vertical angle, obtained from the formula on page 250.

The following programme is that used on the Coast and Geodetic Survey:

Programme for Observing.—Adjust and level instrument; then,

1. Bring the bubble to the center of the graduation and read the micrometer, care being taken to avoid parallax in reading.
2. Bisect the target with the horizontal wire and read micrometer.
3. Reverse level, bisect target, and read micrometer.
4. Bring bubble to the center and read micrometer.
5. Invert telescope, bring bubble to the center, and read micrometer.
6. Bisect target and read micrometer.
7. Reverse level, bisect target, and read micrometer.
8. Bring bubble to the center and read micrometer.

(After beginning the set of observations, all changes in the instrument must be made with the micrometer screw.)

Then the observer notes the readings of the upper and lower edges of the target, and of the distance threads on the painted graduation of the rod. The rodman reads the rod and thermometer, and records them in a book which he carries for the purpose. His reading of the rod is verified by the recorder, and should correspond, within a few millimetres, to the mean obtained from the observer's reading of the upper and lower edge of the target.

During the observations the instrument should be screened from the sun by an umbrella, and also protected from the wind whenever it is strong enough to disturb the level.

Before carrying the instrument forward it should be clamped to the tripod head, the false wyes screwed up to support the telescope, and the cam hook turned to raise the superstructure off the micrometer screw.

The rodmen should exercise great care to plant the foot plates firmly in the ground, and to keep clean the cavities in which the feet of the rods rest.

The rods should be placed at as nearly equal distances from the instrument as possible, and may occasionally be at distances as great

as 150 metres if the atmosphere is very clear and steady, but ordinarily have rarely been much more than 100 metres. The tendency now seems to be to make the sights shorter still, or from 50 to 75 metres on the average.

The above-described method of observing has one weak point, and that is the assumption that the plate carrying the micrometer nut (and all that part of the instrument below it) remains undisturbed while moving the upper part of the instrument, by means of the micrometer screw, from its position for a horizon reading to its position for the reading on the target, or *vice versa.* Those who are using this method, however, consider it superior to any other, and the results obtained show favorably when compared with the best work of this character in existence; thus showing that the errors resulting from the assumption of immobility of the lower part of the instrument while moving the upper are small with skilled observers.

The method of introducing reflecting prisms for reading the level, as used in France, enables the observer to read the rod very quickly after bringing the level bubble to the center. But with this method the rods must be of the self-reading type—i. e., so that the observer reads the rod directly from the instrument; for it is practically impossible for a rodman to set a target accurately in the horizon of the telescope and preserve the verticality of his rod at the same time. By using self-reading rods the sights must be short in order to get accurate readings. Whether the errors in this method due to the inaccuracy of reading the rod and noting the position of the level bubble by means of the reflecting prisms are greater than the assumption of immobility of the lower part of the instrument while moving the upper, as in the Coast and Geodetic Survey method, is still an unsettled question. The members of the Coast and Geodetic Survey, however, consider their method sufficiently more accurate to warrant its use, although it is more laborious and expensive than the French method.

River Crossing.—When a line of precise levels crosses a river it is necessary to make unequal backsights and foresights if the width of the stream exceeds the ordinary distance of the rod from the instrument. The most satisfactory method, used by the Coast and

Geodetic Survey, is to mount an instrument on each side of the river, as nearly in the same horizon as possible, and make a number of simultaneous observations, each observer pointing upon a rod set up near the instrument on the opposite side for the foresight, and upon a rod on a bench near by for the backsight, the same two benches being used by both observers. The observations are made in the same way as in the regular work, taking as many additional sets of observations as may be considered necessary for the condition of the atmosphere—five or six, or more if the atmosphere is unsteady or the river very wide. The observers then change positions and repeat, each with his own instrument. It may be necessary to make observations on different days so as to eliminate bad conditions of the atmosphere as much as possible.

When only one instrument is available observations should be made on both sides of the river, one following the other, and repeated alternately several times as rapidly as possible. By observing in opposite directions the effect of refraction may be supposed to be eliminated, as is that of the earth's curvature.

For good results the line of sight should never pass within ten feet (three metres) of the surface of the water, as the refraction is very uncertain near the surface.

The constants of the instrument must be well determined, especially if two instruments are used, in order to get the best results. The horizontal distances may be obtained by using the micrometer as a gradienter.

749. Bench Marks. The nature and location of bench marks depend upon the purposes they are intended to subserve, and are either permanent or temporary. The latter are used only to afford a check on the leveling operations and as a test of their accuracy They may consist of smooth-headed nails or spikes driven into trees or solid pegs.

Permanent bench marks are intended to secure the results of the work for all time, and are consequently placed where they will remain undisturbed. Public or private buildings, stone piers, outcropping ledges of rock, will usually afford good points of reference. But when these are not available, cut stones, suitably marked,

may be set in the ground for the purpose. When a brick building is to be used, a copper bolt may be set horizontally in a hole drilled in the center of a brick, and adjacent bricks marked.

The frequency of permanent bench marks must depend principally upon the character of the country through which the line passes, but can never be too great. Temporary bench marks should be established at intervals of about a kilometre, to serve as a safeguard against errors and for the comparison of the relative accuracy of the work on different lines.

SPIRIT LEVELING.

Observer, I. W. Instrument, No. 5. Date, July 19th, P. M., and 20th, A. M., 1895.

BACKSIGHT.

No. of station.	Tel. E or I.	Level, D or R.	Micrometer-reading of		Dist. wire, rod reading.	Upper and lower edge of target.	Target, rod reading. Temp. (C.).
			Horizon.	Target.			
			Rod P on B. M. B.				
16	E	D	19·725	19·724			
		R	·750	·725			
	I	R	·750	·729	Tel. E.	1·610	33°
		D	·726	·728	1·010	1·540	1·5718
			19·738	19·727	0·562	1·575	
			−1·1				
			Rod Q on B. M. B.				
16	I	D	19·728	19·711			
		R	·740	·712			
	E	R	·738	·704		1·610	34°
		D	·728	·700	1·013	1·540	1·5747
			19·734	19·707	0·562	1·575	
			−2·7				
			July 20th, A. M., 1895. Rod P on stub *m* established July 19th.				
1	E	D	19·724	19·726			
		R	·720	·726			
	I	R	·721	·728		1·881	24°
		D	·702	·724	2·244	1·811	1·8474
			19·717	19·726	0·397	1·846	
			+0·9				
			Rod Q on stub *n* established July 19th.				
1	I	D	19·702	19·694			
		R	·712	·690			
	E	R	·713	·684		1·880	25°
		D	·702	·688	1·470	1·810	1·8440
			19·707	19·689	0·374	1·845	
			−1·8				

750. Records. The records should be as complete as possible, giving every detail that may have any bearing upon the work, both as to accuracy and location. A minute description should be made of each bench mark, so that it can be readily found by a stranger in the locality; of the times and manner of adjusting the instrument and rods; of the meteorological conditions, particularly the strength of the wind and state of the atmosphere; and of the nature and condition of the ground, whether stable or not, roads, fields, embankments, and the like. The descriptive part of the record can

SPIRIT LEVELING.

Observer, I. W. Instrument, No. 5. Date, July 19th, P. M., and 20th, A. M., 1895.

FORESIGHT.

No. of station.	Tel. E or I.	Level, D or R.	Micrometer-reading of		Dist. wire, rod reading.	Upper and lower edge of target.	Target, rod reading. Temp. (C.).
			Horizon.	Target.			
			Rod P on nail in stub *m*.				
16	E	D	19·730	19·726			
		R	·748	·725			
	I	R	·746	·732		1·310	33°
		D	·726	·733	1·820	1·240	1·2717
			19·738	19·729	0·548	1·275	
			—0·9				
			Rod Q on nail in stub *n*.				
16	I	D	19·738	19·738			
		R	·750	·739			
	E	R	·750	·732		1·298	33°
		D	·740	·730	0·692	1·228	1·2642
			19·745	19·735	0·572	1·263	
			—1·0				
			Fair. Light clouds. Light wind, S.W. Rod P.				
1	E	D	19·738	19·738			
		R	·765	·734			
	I	R	·765	·740		1·235	25·5°
		D	·738	·738	1·600	1·165	1·2003
			19·752	19·738	0·400	1·200	
			—1·4				
			Rod Q.				
1	I	D	19·738	19·715			
		R	·768	·711			
	E	R	·768	·700		1·220	26°
		D	·743	·705	0·802	1·150	1·1853
			19·754	19·708	0·383	1·185	
			—4·6				

not be too complete, in order that a stranger to the work may take up the records and make a thorough discussion of the results.

A specimen double-page record of the observations for a double simultaneous line, as used on the Coast and Geodetic Survey, is given on the preceding pages. The headings of the columns, in connection with the programme for observing on page 255, explain the meaning and origin of the quantities given.

751. Computation. The means of the micrometer readings in the horizon (H.), on the target (T.), and their difference (T. — H.), together with the difference between the readings of the distance wire and rod, are given in the record just below the observations for each set. Then the form of computation, as given below, is filled out, the quantities in columns 1, 1*a*, 2, 2*a*, 4, 4*a*, 9, 9*a*, being taken directly from the record. In the computation the readings on each rod are kept separate, and each computed independently. The combination is made in the abstract of results, given just below the form of computation.

The quantities in columns 3, 3*a*, 5, 5*a*, 6, 6*a*, 7, 7*a*, are obtained from tables made according to the formulas on pages 249–252. A set of such tables must be prepared for each instrument. The following form will usually be found most convenient, as all the corrections are found on one horizontal line for any one set of readings:

ARGUMENT. DISTANCE WIRES.	DIST.	INEQUALITY OF COLLARS	T.—H. IN DIVISIONS OF MICROMETER.									CURVATURE AND REFRACTION
			1^d	2^d	3^d	4^d	5^d	6^d	7^d	8^d	9^d	
	m.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.

FORM 21.—COMPUTATION OF SPIRIT LEVELS.

State, Kansas. Date, 1895. Instrument, No. 5. Rods, P and Q.

BACKSIGHTS.

1	2	3	4	5	6	7	8	9	10	
No. of station.	Distance. Argument.	Distance. Metres.	T.–H.	Corrections. T.–H.	Corrections. Inequality of collars.	Corrections. Curvat're and refraction.	Σ corr.	Rod readings.	Corrected rod readings.	B. M.
	cm.		Div.	mm.	mm.	mm.	mm.	m.	m.	
					Rod P. July 19th, P. M., and 20th, A. M.					B to C
16	56·2	102	–1·1	–1·7	00	–0·7	–2·4	1·5718	1·5694	
1	39·7	72	+0·9	+1·0	00	–0·3	+0·7	1·8474	1·8481	
*2	15·7	29	–1·1	–0·4	00	–0·1	–0·5	1·6594	1·6589	
		203		–1·1		–1·1	–2·2	5·0786	5·0764	
					Rod Q. July 19th, P. M., and 20th, A. M.					B to C
16	56·2	102	–2·7	–4·2	00	–0·7	–4·9	1·5747	1·5698	
1	37·4	68	–1·8	–1·8	00	–0·3	–2·1	1·8440	1·8419	
*2	14·2	26	–1·2	–0·5	00	–0·0	–0·5	1·6417	1·6412	
		196		–6·5		–1·0	–7·5	5·0604	5·0529	
	*		*		*		*		*	*

FORESIGHTS.

1a	2a	3a	4a	5a	6a	7a	8a	9a	10a	
No. of station.	Distance. Argument.	Distance. Metres.	T.–H.	Corrections. T.–H.	Corrections. Inequality of collars.	Corrections. Curvat're and refraction.	Σ corr.	Rod readings.	Corrected rod readings.	BS.-FS.
	cm.		Div.	mm.	mm.	mm.	mm.	m.	m.	m.
					Rod P. July 19th, P. M., and 20th, A. M.					
16	54·8	100	–0·9	–1·2	00	–0·7	–1·9	1·2717	1·2698	
*1	40·0	73	–1·4	–1·5	00	–0·4	–1·9	1·2003	1·1984	
2	16·2	30	–5·4	–2·5	00	–0·1	–2·6	0·1722	0·1696	
		203		–5·2		–1·2	–6·4	2·6442	2·6378	+2·4386
							Temperature correction,			0·0000
					Rod Q. July 19th, P. M., and 20th, A. M.					
16	57·2	104	–1·0	–1·6	00	–0·7	–2·3	1·2642	1·2619	
*1	38·3	70	–4·6	–4·8	00	–0·3	–5·1	1·1853	1·1802	
2	16·2	30	–3·8	–1·8	00	–0·1	–1·9	0·1722	0·1703	
		204		–8·2		–1·1	–9·3	2·6217	2·6124	+2·4405
							Temperature correction,			0·0000
*		*		*		*		*	*	

* Distances indirect as compared with foresight of Station 1.

FORM 22.—ABSTRACT OF SPIRIT-LEVEL RESULTS.

State, Kansas. Instrument, No. 5. Rods, P and Q. Observer, I. W. Computer, I. W.

DATE.	From B. M. to B. M.	Distance in kilometres.	DIFFERENCE OF HEIGHT. Line. Rod P.	Line. Rod Q.	Mean.	DISCREPANCY. Partial. P–Q.	Total accumulated.	No. of B. M.	Dist'nce from B. M. No. B.	Height above B. M. B.	Locality.
			m.	m.	m.	mm.	mm.		km.	m.	
1895								B	0·0		Mark on bridge near Desoto.
July 19–20	B to C	0·406	+2·4386	+2·4405	+2·4395	−1·9	−1·9	C	0·4	+2·4395	Bolt at Desoto.
20	C " 17	0·948	−1·6894	−1·6862	−1·6878	−3·2	−5·1	17	1·3	+0·7517	
20	17 " 18	1·271	+0·3144	+0·3139	+0·3142	+0·5	−4·6	18	2·6	+1·0659	
20–22	18 " 19	1·160	−1·1657	−1·1621	−1·1639	−3·6	−8·2	19	3·8	−0·0980	
22	19 " 20	1·034	+1·1083	+1·1046	+1·1064	+3·7	−4·5	20	4·8	+1·0084	
22	20 " 21	1·001	+1·5439	+1·5439	+1·5439	+0·0	−4·5	21	5·8	+2·5523	

752. Curvature and Refraction. The datum surface to which all elevations are referred is that surface of the earth which corresponds to the mean sea level. As this surface is curved, any level line must be curved to the same extent as that part of the surface over which it passes. Hence a line of sight which is horizontal at the point where the instrument is located will strike the rod at a point above the true level line by an amount equal to the curvature of the earth's surface for that distance, disregarding refraction. As the lines of sight are always very short, we may compute the correction for curvature (c) from the formula, $c = s \sin. \frac{1}{2}\psi$, where s is the length of the line of sight, and ψ the angle at the center of the earth subtended by the arc s. But $\sin. \psi = \frac{s}{\rho}$, whence $c = \frac{s^2}{2\rho}$, which is accurate enough for the short distances used in precise leveling.

A greater source of error, however, is the refraction of the line of sight while passing through the atmosphere. This refraction is constantly changing, and at such elevations as are necessary in precise leveling is a

particularly variable quantity. Over water it is usually very large, while over dry sandy regions or from elevated stations it is small. Its effect is the reverse to that of curvature, and has been found by experience to be about one sixth to one eighth of the latter. A table may then be computed giving the correction, for the argument s, necessary to reduce the actual reading of the rod to what it should read for the true level line, from the formula

$$c = -\frac{7 s^2}{16 \rho}$$

using the mean value for the radius of curvature, $\rho = 6{,}370{,}000$ metres. A convenient approximate formula is

$$c = -\frac{11}{16} \quad \frac{s^2}{10\,000}$$

where s is in metres and c in millimetres.

The effects of curvature and refraction are very nearly eliminated by making the backsights equal the foresights as closely as possible, usually within a metre or two. This is always done on the best class of work.

753. Temperature Correction. The effect of temperature on the length of the rod used in leveling is always a small quantity, hence we may determine its effect from one bench mark to the next, and apply a correction to the difference in height of the two bench marks. A convenient formula for the purpose is

$$c = [r]\,[t]\,k - [r']\,[t']\,k$$

where $[r\,]$ is the sum of all the rod readings on the backsights;

$[r']$ is the sum of all the rod readings on the foresights;

$[t\,]$ is the sum of all the temperature readings on the backsights after subtracting the temperature at which the rod is standard;

$[t']$ is the sum of the corresponding temperature differences for the foresight readings; and

k is the coefficient of expansion of the rod.

Since the temperature at the foresight reading rarely differs much from that at the backsight reading, we may put $[t] = [t']$ and get

$$c = ([r] - [r'])\,\tfrac{1}{2}\,([t] + [t'])\,k$$

This formula is applicable to any rod if the scale is fastened at the foot and free to expand upward. When a metal scale with a large coefficient of expansion is used, and the t's are large and irregular, it may be necessary to use the more rigid formula developed first.

754. Index Correction of the rod must be applied whenever the number of backsights on the rod differs from the number of foresights. The effect is eliminated whenever it is used the same number of times for each. For example, if the backsight is on the rod and the foresight on a scale, the backsight reading of the rod must be corrected for its index correction.

755. Additional Corrections. Owing to the irregularity in the shape of the earth, together with the fact that it is not a sphere, several other errors are introduced in the determination of the difference of height between two stations some distance apart.

1. In running a so-called level line we must depend upon the vertical to the surface of the earth for each position of the instrument, as shown by the level on the instrument. But these local verticals are more or less deflected from the true vertical by the attraction of preponderating earth masses of the region, such as mountains, hills, etc. Hence the line we actually run is a very irregular one, and not the smooth curve we would obtain if the verticals all belonged to a figure of revolution. If we run a line of levels up the side of a mountain or hill the vertical is inclined the same direction all the way, and the elevation obtained is consequently very much in error, the measured difference in height always being too small, unless there is some stronger attracting force outside of the mountain than within, so as to throw the vertical in the opposite direction. It is practically impossible to entirely eliminate or correct the errors arising from this source. If a mountain is large, we may get a rough value for its mass and center of gravity, and then compute the deflections of the verticals, and obtain the correction to the observed difference in height in order to get the true difference, but this is a very unsatisfactory and difficult process, and rarely, if ever, used. The only practical way is

to run the lines in such a manner as to make the effect as small as possible.

From the laws of attraction it is evident that the steeper the slope of the mountain the greater will be the inclination of the vertical, and therefore the greater the error in the determination of the difference in height; hence, if a mountain range is crossed by ascending over a gentle slope and descending by a sharp slope, we will not obtain the same difference in height, and consequently all the elevations on the other side will be affected to the extent of this difference.

2. This correction is due to the shape and motion of the earth, and is one that can be determined with a fair degree of accuracy.

In shape we know that the earth approximates very closely to an oblate spheroid or ellipsoid of revolution. The dimensions of this spheroid, as adopted in the United States for the reduction of observations, are those of Gen. A. R. Clarke, published in 1866, viz.:

Semimajor axis, $a = 20,926,062$ English feet, log. $= 7{\cdot}3206875$
Semiminor axis, $b = 20,855,121$ " " log. $= 7{\cdot}3192127$

Or

$a = 6,378,276$ metres, log. $= 6{\cdot}8047033$
$b = 6,356,652$ " log. $= 6{\cdot}8032285$

Since the earth is considered as a mobile mass rotating about an axis, its equipotential surfaces are not parallel to each other, but are closer together on the axis of rotation and farther apart at right angles to it. If two points on an equipotential surface are connected by running a line of levels between them along this surface, they will be found to be of equal height, since the verticals are the normals of this surface; but the datum surface to which all elevations on the earth's surface are referred is the equipotential surface of mean sea level, hence a line of levels run on any other equipotential surface gives a difference of elevation which is erroneous to the amount of the convergence of these two surfaces in the distance run. Elevations corrected for this convergence are called *orthometric heights*, and are the actual elevations above the datum surface, mean sea level.

The formula for deriving this correction is obtained as follows

(see Jordan's "Vermessungskunde," pp. 466–470; also Helmert's "Höheren Geodäsie," pp. 241, 609):

If g_o is the acceleration of gravity at the surface of mean sea level, g its value at a height h above, then from the law of gravitation

$$\frac{g}{g_o} = \frac{\rho^2}{(\rho + h)^2} = 1 - \frac{2h}{\rho} + \text{etc.} \qquad [50.]$$

ρ being the radius of curvature of the earth at that point.

If G_o be the sea-level value of gravity at the equator, and

G'_o the sea-level value of gravity at the pole,

m the ratio of the centrifugal force at the equator to the force of gravity,

e the ellipticity of the earth—i. e., $\frac{a-b}{a}$;

then from Clairaut's theorem we have

$$\frac{G'_o - G_o}{G_o} = \tfrac{5}{2}m - e = \beta \,(= 0{\cdot}0052, \text{ practically}).$$

Since the value of gravity varies in proportion to the sin.$^2\,\phi$ (latitude), we have

$$g_o = G_o\,(1 + \beta \sin.^2 \phi) \qquad [51.]$$

Substituting this value for g_o in [50], we get

$$g = G_o\,(1 + \beta \sin.^2 \phi)\left(1 - \frac{2h}{\rho}\right) \qquad [52.]$$

Suppose, now, we have two equipotential surfaces (such as the sea-level surface of the earth and a surface along a tableland may be considered), distant from each other h' at latitude ϕ', and h'' at ϕ'', and g'_o, g', and g''_o, g'', the corresponding values of gravity at the sea-level surface and tableland surface at the two latitudes; then, from the law of equipotential surfaces, $g'h' = g''h''$ = a constant for these two surfaces, and equation [52] may be written

$$\frac{g'}{g''} = \frac{h''}{h'} = \frac{G_o\,(1 + \beta \sin.^2 \phi')\left(1 - \frac{2h'}{\rho}\right)}{G_o\,(1 + \beta \sin.^2 \phi'')\left(1 - \frac{2h''}{\rho}\right)}$$

or

$$\frac{h''}{h'} = \frac{1 + \beta \sin.^2 \phi'}{1 + \beta \sin.^2 \phi''} \qquad [53.]$$

since h' and h'' are so nearly equal that the terms $1 - \frac{2h'}{\rho}$ and

$1 - \frac{2h''}{\rho}$ are practically identical. Clearing equation [53] from fractions and reducing, we have

$$h' - h'' = \beta h \sin.(\phi'' + \phi') \sin.(\phi'' - \phi') \qquad [54.]$$

which gives the convergence of two equipotential surfaces for the difference in latitude $(\phi'' - \phi')$, the average distance between the two surfaces being h. It is therefore the correction that must be applied to an elevation obtained by leveling along an equipotential surface, at a height h above the datum surface, in order to obtain the actual elevation of the point referred to the datum surface—i. e., its orthometric height.

When the line of levels passes over irregular surfaces it may be divided into short sections, and the orthometric corrections obtained for each section by using the mean equipotential surface in that section.

If the difference in latitude is not very great, we may put $\sin.(\phi'' - \phi') = \frac{s}{\rho}$, where s is the meridian distance between the parallels of latitude of the two stations. Then, with $\beta = 0{\cdot}0052$, $\rho = 6{,}370{,}000$ metres, and taking s in kilometres,

$$h' - h'' \text{ (in millimetres)} = 0{\cdot}00082\, h\, s \sin.(\phi'' + \phi') \qquad [55.]$$

As an example, take the line of levels from Biloxi, on the Gulf of Mexico ($\phi' = 30° 24'$), to Odin, Ill. ($\phi'' = 38° 38'$), where $h = 126$ metres. Assuming that the slope is uniform from the Gulf to Odin, we determine the convergence of the two equipotential surfaces, the mean sea-level surface and the equipotential surface 63 metres (one half of 126) above it, from equation [54]. The result is 44 millimetres. Hence the elevation of Odin, as obtained from the observations, must be decreased 44 millimetres in order to get the orthometric height, or actual elevation, above the surface of mean sea level.

The effect on a line of levels of the attraction of the sun and moon upon the earth's crust has been discussed, but is too small to be appreciable, and is probably more or less compensating in a long line.

756. Precision of a Line of Levels. The precision of a line of levels is given by the probable error or mean error of a sin-

gle leveling of a unit distance, which is usually taken as a kilometre. If we suppose the conditions equally favorable the whole length of the line, and that none but accidental errors may be expected to enter, then if μ is the mean error of the unit of distance (1 kilometre), the mean error of a distance s kilometres would be $\mu\sqrt{s}$—i. e., the weight of a leveling of a distance of s kilometres is inversely proportional to that distance, since the weight of an observation is inversely proportional to the square of its mean error. In order to be exact, we should combine the mean errors arising from all the sources of error that enter into the work (such as errors of standardization of the rods, character of the country leveled over, etc.), considering each as arising from independent sources. Then we should find the m. e. (mean error) of each line leveled to be $\sqrt{[\mu^2]}$, and the weight of the line therefore proportional to $\frac{1}{[\mu^2]}$. As the influence of the distance so largely exceeds the effect of these other errors, it is sufficient to adopt the rule first given of weighting as the inverse distance.

It follows from this rule that better results are to be expected if short sights are taken; and experience has shown that 100-metre sights are about as long as can be used on the best work.

Suppose we have a line of levels s kilometres in length, over which n_1 independent levelings have been made, and that the results have been compared at intermediate bench marks in succession s_1, $s_2 \ldots . s_n$ kilometres apart. Then, if v_1', $v_1'' \ldots . v_1^{n}$; v_2', $v_2 \ldots . v_2^{n} \ldots .$ represent the residual errors of the differences in height in the n sections, and p_1, $p_1'' \ldots .$; p_2, $p_2' \ldots .$; $\ldots .$ denote the weights of the observed differences, we have by the principles of least squares,

1. On the hypothesis that the precision for each unit of distance (1 kilometre) is the same throughout the different sections,

$$\mu^2 = \frac{[p\,v\,v]}{n\,(n_1 - 1)}$$

or, since the weights are inversely proportional to the distances,

$$\mu^2 = \frac{1}{n\,(n_1 - 1)} \left[\frac{v\,v}{s}\right] \qquad [56.]$$

2. On the hypothesis that the sections are independent, the average mean error is given by

$$\mu^2 = \frac{n_1}{n_1 - 1}\Big([v_1 v_1] + [v_2 v_2] + \ldots . [v_n v_n]\Big)\frac{1}{[s]} \qquad [57.]$$

When the number of levelings is two, $n_1 = 2$. If $d_1, d_2 \ldots . d_n$ are the differences in the respective sections (corresponding to $2\,v'_1$, $2\,v'_2 \ldots . 2\,v'_n$ above), these formulas become respectively

$$\frac{1}{2\,n}\left[\frac{d\,d}{s}\right] \quad \text{and} \quad \tfrac{1}{2}\frac{[d\,d]}{[s]} \qquad [56\,\text{a.}]\ [57\,\text{a.}]$$

The mean error (μ_n) of a single leveling of the whole line for n_1 levelings would be, for the above hypotheses respectively,

$$\mu^2_n = \frac{[s]}{n\,(n_1 - 1)}\left[\frac{v\,v}{s}\right] \quad \text{and} \quad \frac{n_1}{n_1 - 1}\left([v_1 v_1] + [v_2 v_2] + \ldots .\right)$$

or, for two levelings,

$$\mu^2_2 = \frac{[s]}{2\,n}\left[\frac{d\,d}{s}\right] \quad \text{and} \quad \frac{[d\,d]}{2}$$

The mean error of the mean of two measures is obtained by dividing μ_2 by $\sqrt{2}$.

In order to get the probable error, multiply the mean error by 0·6745.

As a convenient guide for the field observer, the difference d between two determinations of the elevation of a station should always be less than 5 millimetres $\sqrt{s}$, where s is the distance in kilometres from the point of beginning. This rule is used on the Coast and Geodetic Survey, and also by several other bodies, where the highest degree of precision is desired.

757. Adjustment of Levels. When but one line of levels is to be run, it is necessary to make at least two levelings of this line in order to have some criterion as to the accuracy of the work. If comparisons are made at intermediate points not too far apart, and the discrepancies found do not exceed the limits mentioned in the preceding paragraph, then the mean of the results may be taken as giving the elevations sought, after the correction for reduction to orthometric heights has been applied, as also all other corrections that can be determined.

But when a complete survey of a country is made for topo-

graphic, geodetic, or engineering purposes, and a network of lines of levels is run, an additional control of the accuracy of the work is afforded by the polygonal closing of the level lines forming each figure—i. e., from the condition that if we measure all around a polygon we should obtain the same elevation of the initial point as that with which we started, or the sum of the differences of height of all the lines forming the polygon, taken consecutively, should be zero, provided these differences of height are orthometric. This is on the assumption that the errors of closure depend upon the errors of observation only, which is usually the case when the net is small and the country free from local deflections of the vertical. As it is practically impossible, with our present knowledge of the earth, to either determine the effect of the deflections of the vertical or to eliminate it, it must appear in the closing errors to a certain extent. All other systematic and accidental errors must also appear in the closing errors.

There are several methods of adjusting a net of levels—i. e., distributing the small discrepancies that are usually developed. The rigid adjustment by the method of least squares may be carried out in a way similar to that used in triangulation, the conditions to be satisfied among the observed differences of height being of a similar character:

1. Those due to the nonagreement of repeated measures of the difference in height between successive bench marks which correspond to the station conditions of triangulation; and

2. Those due to the condition that the sum of all the differences of height in any closed polygon should be zero. This corresponds to the figure adjustment of triangulation.

As in triangulation, it is usually accurate enough to separate the two sets of equations, first adjusting the local or bench mark to bench-mark conditions, and afterward satisfying the conditions of closure, using the values of the various lines of the polygons that have been freed from the local conditions.

In order to be certain that we have formed all the closure conditions, and have used no more than is necessary, we can easily estimate the number required, since in any net the number of lines (l) necessary to fix s bench marks is $s - l$; hence the number of super-

fluous lines—i. e., the number of closure conditions to be satisfied among the differences of height—is

$$l - s + 1$$

For examples and further information, see Wright's "Adjustment of Observations," Jordan's "Vermessungskunde," etc.

Approximate Method.—A very simple approximate method of adjusting a net of levels is to distribute the errors of closure among the sides of the polygons in proportion to the square root of the length of the sides, beginning with the figure that has the largest closing error, then the next largest, and so on. When a line already adjusted forms a side of another figure, its adjusted value is used and the whole error distributed among the remaining sides. Experience has shown that the errors of a line of levels are partly compensating, and that they are more nearly proportional to the square root of the distance than the square, which is the reason for the above method of distributing the errors.

758. Mean Sea Level. The determination of the datum surface for heights, mean sea level, is by no means an easy problem, owing to the variation of the tides, irregularities in the shape of the earth, ocean currents, storms, and, in fact, everything which may have any effect upon the level of the sea.

Since the tides are due primarily to the attraction of the moon, and secondarily to that of the sun, it is evident that observations must be made during at least a lunar month in order to obtain a result that will be of any value whatever. In order to get an average value for the effect of the sun's attraction, it is also evident that the observations must be continued for at least a year. This, however, gives but one result for each of the various positions of the sun and moon, and part of those may be abnormal owing to the effect of storms or other causes. Hence, in order to obtain an accurate value of the mean sea level, continuous observations for several years are made with automatic self-registering tide gauges.

An automatic self-registering tide gauge is simply a device for recording automatically the relative heights of a float on the surface of the water, this float moving freely up and down in a perforated box so as to be free from the action of the waves. The float is con-

nected by appropriate mechanism with a pencil which moves up and down over a sheet of paper kept in uniform horizontal motion by means of clockwork. The vertical motion of the pencil is directly proportional to the vertical motion of the float, so that the curve it traces is proportional to the tidal motion. The record thus made is referred to the absolute height by taking readings of the height of the water on a staff gauge (which must be connected with the automatic gauge) and noting these readings on the sheet together with the time.

This outer staff must be connected by careful leveling with fixed bench marks in the immediate vicinity, and its stability determined by frequently repeated levelings.

In order now to obtain the height of mean tide for a long period ordinates are measured from a fixed or datum line of the sheet for each hour during the entire period, provided this period is composed of complete lunar months, and the mean of all these hourly ordinates is taken as the mean tide. It is checked by taking the mean of all the daily maxima and minima for the same period. It may be found necessary to reject some lunations owing to abnormal results during storms, etc.; only complete lunations should be rejected, however.

If the best possible results are desired, the tide gauge must be placed where it is free from currents and the piling up of water due to narrow and shallow channels—i. e., it should record the height of the water as it is in the open ocean.

As an instance of the variation in the tides, take the observations at St. Augustine, Fla., where the average rise and fall is less than 5 feet, yet the mean of the hourly ordinates for two different lunar months differ as much as 1·32 feet, and the mean of the hourly ordinates on one day differs from that of another 3·2 feet, thus showing the uncertainty of using short periods for the determinations of mean sea level. After correcting the lunations for the annual variation—i. e., effect of the sun—the range in the values for lunations is only 0·43 feet, which is due almost entirely to local causes, storms, etc.

CHAPTER XV.

BAROMETRIC LEVELING.

759. Principles. The difference of the heights of two places may be determined by finding the difference of their depths below the top of the atmosphere, in the same way as the comparative heights of ground under water are determined by the difference of the depths below the top of the water. The desired height of the atmosphere above any point, such as the top of a mountain or the bottom of a valley, is determined by weighing it. This is done by trying how high a column of mercury or other liquid the column of air above it will balance, or what pressure it will exert against an elastic box containing a vacuum, etc. Such instruments are called *barometers.*

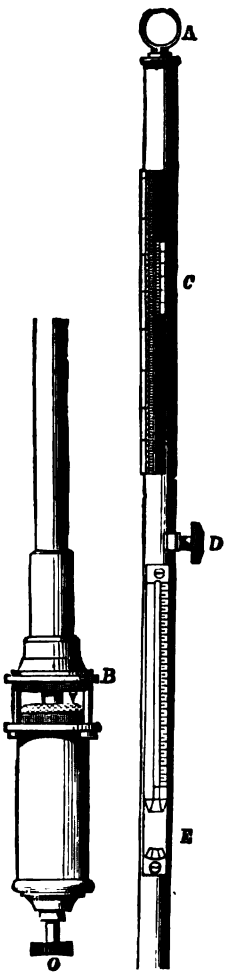
If at the level of the sea a glass tube 32 or 33 inches long, closed at one end and filled with mercury, be placed vertically, with its open end immersed in a vessel of mercury, the mercury in the tube will fall to about 30 inches in height. This is the height of a column of mercury which the pressure of the atmosphere will balance at the sea level. If the tube and vessel of mercury be inclosed in a case so as to be portable, and furnished with a scale and vernier to enable the height of the mercury in the tube to be accurately read, it becomes a barometer.

Instruments.

760. Barometers made for leveling are called *mountain barometers.* Of these there are several varieties. The one shown in Fig. 482 is a cistern barometer. This consists of a column of mercury contained in a glass tube about 32 inches long, closed at the top, and whose lower end is placed in a cistern of mercury. The tube is covered with a brass case, terminating at the top in a ring, A, for suspension, and at the bottom in a flange, B, to which the

cistern is attached. At C is a vernier, by which the height of the mercury is read off. The vernier is moved by means of a rack worked by the milled head shown at D. The zero of the scale is a small ivory point, shown below the flange B. The upper part of the cistern is of glass, so that the surface of the mercury in the cistern and the ivory point may be readily seen. At E is the attached thermometer which indicates the temperature of the mercury. Below the glass the cistern is of boxwood, with a bottom of buckskin resting on a metal plate, against which the milled-headed screw O presses. The boxwood cistern with its adjustable bottom is inclosed in a brass case attached to the flange B. The mercury in the cistern is raised or lowered, by means of the milled-headed screw O, till its surface is just in contact with the ivory point. When a reading is to be taken the instrument should hang by the ring, A, so as to swing freely.

Fig. 482.



When the barometer is to be taken from one station to another it is placed in a case fitted to it, and should be carried with the cistern up.

Repairs.—When the barometer is to be used in a locality where it can not easily be returned to the maker for repairs, the materials for making repairs should be provided. These are, extra tubes, pure mercury, chamois skin, kid, shoemakers' thread, and wax. If a tube be broken, the instrument should be taken apart and the mercury poured into a clean vessel. In taking the instrument apart note carefully how the parts are connected, and especially how the leather collar of the cistern is tied on the tube, so that all the parts may be properly replaced.

The place for breaking off the end of the new tube will be de-

termined by inspecting the old one. The break is made by first cutting the tube with a sharp file. The mercury for filling the tube should be strained through chamois skin, and poured into the tube through a glass or paper funnel, filling the tube within about an inch of the top. Cover the finger with kid and place it over the end of the tube, and let the bubble of air travel slowly up and down the tube, collecting all of the bubbles. When the mercury is free from bubbles, fill the tube to the top and invert it in the cistern. The finger used in closing the open end of the tube, in inserting the tube in the cistern, or in taking it out, should be covered with kid. A better method of filling the tube is to boil the mercury in the tube during the process of filling. Three or four inches of mercury are placed in the tube at a time and boiled over a spirit lamp. This process is repeated until the tube is filled. The mercury to fill the cistern is also boiled.

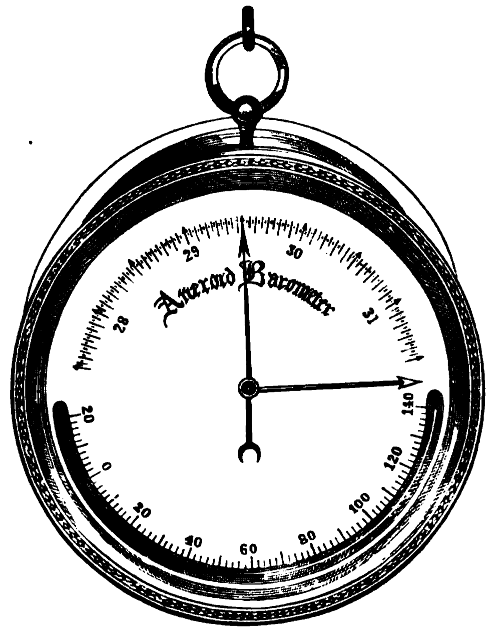
When the mercury in the cistern becomes dirty it should be taken out and strained through chamois skin, and then replaced in the cistern.

761. The Siphon Barometer. In the siphon barometer, instead of inserting the open end of the tube of mercury in a cistern of mercury, the tube is bent up at the lower end, making a second vertical branch about one quarter as long as the main branch. This second branch is closed at the top, and a small orifice is made in the side of the tube a short distance from the end to admit air. The air being exhausted from the long branch and admitted to the short one, the difference between the heights of the mercury in the two branches gives the height of the column of mercury due to atmospheric pressure.

762. The Aneroid Barometer. This is a thin box of corrugated metal, exhausted of air. When the air grows heavier the box is compressed, and when the air grows lighter it is expanded by a spring inside. This motion is communicated by suitable levers to the index hand on the face, which indicates the pressure of the atmosphere, the face being graduated to correspond with a common barometer.

There are several varieties of this instrument, differing principally in the method of determining the movement of the corrugated box due to changes in the density of the atmosphere. They are made in sizes varying from two to six inches in diameter. They

FIG. 483.



are much used on account of their portability, but are not as reliable as the mercurial barometer.

Approximately, a difference of reading of $\frac{1}{100}$ of an inch corresponds to a difference of height of nine feet. The following table is more nearly accurate:

MEAN TEMPERATURE.	30°	40°	50°	60°	70°	80°
Mean pressure, 27 inches...........	9·7	9·9	10·1	10·3	10·5	10·8
" 28 "	9·3	9·5	9·8	10·0	10·2	10·4
" 29 "	9·0	9·2	9·4	9·6	9·8	10·0
" 30 "	8·7	8·9	9·1	9·3	9·5	9·7

763. The Hypsometer. The temperature at which water boils varies with the pressure of the atmosphere, and therefore decreases

in ascending heights. The hypsometer is a thermometer with convenient attachments for determining the boiling point of water. The thermometer then becomes a substitute for a barometer.

Lefroy's approximate rule is: Calling the boiling point at the sea level 212° F., allow 511 feet for the first degree of difference, and increase the number to be added two feet for each degree. Then we have:

Temperature of Boiling Water.	Corresponding Height.	Temperature of Boiling Water.	Corresponding Height.
212°......	0	208°......	1539′ + 517′ = 2056′
211°......	511′	207°......	2056′ + 519′ = 2575′
210°......	511′ + 513′ = 1024′	206°......	2575′ + 521′ = 3096′
209°......	1024′ + 515′ = 1539′	etc.	etc.

For minute hypsometric table see Guyot's Tables.

764. Applications. *Use of the Mercurial Barometer.*—Since the column of mercury in the barometer tube is supported by the column of air above the barometer, the mercury sinks when the barometer is carried higher, and *vice versa.*

The weight of any portion of air decreases from the surface of the earth to the assumed surface of the atmosphere. It has been found by experiment that as the heights to which the barometer is carried increase in arithmetical progression, the weights of the column of air above the barometer, and consequently its readings, decrease in geometrical progression, and that the difference of the heights of any two not very distant points on the earth's surface is equal to the difference of the logarithms of the readings of the barometer at those points, multiplied by some constant coefficient. This coefficient is found by experiment to be 60158·6 at the freezing point, or temperature of 32° F., the readings of the mercury being in inches, and the product, which is the difference of height, being in feet.

Several corrections are necessary.

765. Correction for Temperature of the Mercury. If the temperature of the mercury be different at the two stations, it is expanded at the one and contracted at the other to a height

different from that which is due to the mere weight of the air above it.

Mercury expands about $\frac{1}{10000}$ of its bulk for each degree of F. Therefore this fraction of the height of the column is to be added to the height of the colder column, or subtracted from the height of the warmer one, in order to reduce them to the same standard. A thermometer is therefore attached to the instrument in such a manner as to give the temperature of the mercury.

If a brass scale be used, as is usually the case, the expansion of the brass will be 0·0000104 for each degree F., and the difference between the expansion of the mercury and the brass—the proper correction—will be 0·0000896 of the height of the column for each degree F.

766. Correction for Temperature of the Air. The warmer the air is, the lighter it is; so that a column of warm air of any height will weigh less than when it is colder. Consequently, the mercury in warm air falls less in ascending any height, and is higher at the place than it otherwise would be. Hence the height given by the preceding approximate result will be too small, and must be increased by $\frac{1}{480}$ part for each degree F. that the temperature of the air is above 32°.

767. Other Corrections. For *very* great accuracy, we should allow for the variation of gravity corresponding to the variation of latitude on either side of the mean. So, too, we should allow for the decrease of gravity corresponding to any increase of height of the place.

768. Guyot's Formula and Tables. Many formulas have been proposed for determining heights by the barometer. The one most used in the United States is that devised by Laplace, and published in English measures, with tables by Guyot, in the "Smithsonian Miscellaneous Collection," vol. i, from which the formula and tables here given are taken:

ARRANGEMENT OF THE TABLES.

If we call

$$\left.\begin{array}{l} h = \text{the observed height of the barometer} \\ \tau = \text{the temperature of the barometer} \\ t = \text{the temperature of the air} \end{array}\right\} \text{at the lower station;}$$

$$\left.\begin{array}{l} h' = \text{the observed height of the barometer} \\ \tau' = \text{the temperature of the barometer} \\ t' = \text{the temperature of the air} \end{array}\right\} \text{at the upper station.}$$

If we make, further,

Z = the difference of level between the two barometers;

L = the mean latitude between the two stations;

H = the height of the barometer at the upper station reduced to the temperature of the barometer at the lower station; or,

$H = h' \{1 + 0{\cdot}00008967\,(\tau - \tau')\}$;

The expansion of the mercurial column, measured by a brass scale, for 1° F. = 0·00008967;

The increase of gravity from the equator to the poles = 0·00520048, or 0·00260 to the 45th degree of latitude;

The earth's mean radius = 20,886,860 English feet;

Then Laplace's formula, reduced to English measures, reads as follows:

$$Z = \log.\frac{h}{H} \times 60158{\cdot}6 \text{ English feet} \begin{cases} \left(1 + \dfrac{t + t' - 64}{900}\right) \\ (1 + 0{\cdot}00260 \cos. 2\,L) \qquad [1.] \\ \left(1 + \dfrac{z + 52252}{20886860} + \dfrac{h}{10443430}\right) \end{cases}$$

Table I gives, in English feet, the value of log. H or $h \times 60158{\cdot}6$ for every hundredth of an inch, from 12 to 31 inches in the barometer, together with the value of the additional thousandths, in a separate column. These values have been diminished by a constant, which does not alter the difference required.

Table II gives the correction 2·343 feet $\times (\tau - \tau')$ for the difference of the temperatures of the barometers at the two stations, or $\tau - \tau'$. As the temperature at the upper station is generally lower, $\tau - \tau'$ is usually positive, and the correction *negative*. It becomes *positive* when the temperature of the upper barometer is higher, and $\tau - \tau'$ negative. When the heights of the barometers have been

reduced to the same temperature, or to the freezing point, this table will not be used.

Table IV shows the correction $D' \frac{z + 52252}{20886860}$ to be applied to the approximate altitude for the decrease of gravity on a vertical acting on the density of the mercurial column. It is always *additive.*

Table V furnishes the small correction $\frac{h}{10443430}$ for the decrease of gravity on a vertical acting on the density of the air, the height of the barometer h at the lower station representing its approximate altitude. Like the preceding correction, it is always *additive.*

Use of the Tables.

In Table I find first the numbers corresponding to the observed heights of the barometer h and h'. Suppose, for instance, $h = 29{\cdot}345$ inches; find in the first column on the left the number 29·3; on the same horizontal line, in the column headed ·04, is given the number corresponding to $29{\cdot}34 = 28121{\cdot}7$; in the last column but one on the right, we find for $\cdot 005 = 4{\cdot}5$, or for $29{\cdot}345 = 28126{\cdot}2$. Take likewise the value of h', and find the difference.

If the barometrical heights have not been previously reduced to the same temperature, or to the freezing point, apply to the difference the correction found in Table II opposite the number representing $\tau - \tau'$; we thus obtain the approximate difference of level, D.

For computing the correction due to the expansion of the air according to its temperature, or $D \times \left(\frac{t + t' - 64}{900}\right)$, make the sum of the temperatures, subtract from that sum 64; multiply the rest into the approximate difference D, and divide the product by 900. This correction is of the same sign as $(t + t' - 64)$. By applying it, we obtain a second approximate difference of level, D'.

In Table III, with D' and the mean latitude of the stations, find the correction for variation of gravity in latitude, and add it to D', paying due attention to the sign.

In Table IV with D′, and in Table V with D′ and the height of the barometer at the lower station, take the corrections for the decrease of gravity on a vertical, and add them to the approximate difference of level.

The sum thus found is the true difference of level between the two stations, or Z; by adding the elevation of the lower station above the level of the sea, when known, we obtain the *altitude* of the upper station.

By the use of the small table, VI, approximate differences of level can be obtained by a single multiplication.

Example 1. Measurement of Mount Washington, New Hampshire, by A. Guyot, August 8, 1851, 4 P. M.; the barometer at the lower station being at 825 English feet above the mean level of the sea; at the upper station at one foot below the summit.

The observation gave

	Barometer.	Attached Thermometer.	Temperature of Air.
Gorham...........	$h = 29{\cdot}272$ in.	$\tau' = 70{\cdot}70°$ F.	$t = 72{\cdot}05°$ F.
Mount Washington,	$h' = 24{\cdot}030$ "	$\tau' = 54{\cdot}52°$ F.	$t' = 50{\cdot}54°$ F.
		$\tau - \tau' = 16{\cdot}38°$ F.	$122{\cdot}59°$ F.
			$- 64°$
		$t + t' - 64 =$	$58{\cdot}59°$ F.

Table I gives for $h = 29{\cdot}272$ inches.......	28,061·00
" " for $h' = 24{\cdot}030$ "	22,905·60
Difference....	5,155·40
Table II gives for $\tau - \tau' = 16{\cdot}38°$.........	− 37·64
Approximate difference of level, D′ =	5,117·76
$\dfrac{D \times (t + t' - 64)}{900} = \dfrac{5118 \times 58{\cdot}6}{900} =$	333·19
Second approximate difference, D′ =	5,450·95
Table III gives for D′ = 5450 and lat. 44°	0·50
" IV " for D′ = 5450	14·94
" V " for $h' = 29{\cdot}27$.	0·00
Barometer below summit ..	− 1·00
Mount Washington above Gorham, or Z =	5,465·39
Barometer at Gorham above sea level.....	825·00
Mount Washington above the sea, or altitude	6,290·39 Eng. ft.

Example 2. Measurement of the highest peak of the Black Mountain, in North Carolina, July 11, 1856, by A. Guyot.

By observation we have at

	Barometer.	Attached Thermometer.	Temperature of Air.
Mountain House....	$h = 24{\cdot}934$ in.	$\tau = 64{\cdot}58°$ F.	$t = 61{\cdot}34°$ F.
Highest peak.......	$h' = 23{\cdot}662$ "	$\tau' = 61{\cdot}88°$ F.	$t' = 59{\cdot}36°$ F.
		$\tau - \tau' = 2{\cdot}70°$ F.	$120{\cdot}70°$ F.
			$- 64°$
			$t + t' - 64 = 56{\cdot}7°$ F.

Table I gives for $h' = 24{\cdot}934$............. 23,870·4
" " for $h' = 23{\cdot}662$............. 22,502·4
Difference......... 1,368·0
Table II gives for $\tau - \tau' = 2{\cdot}7$............ − 6·3
Approximate difference, D = 1,361·7

$$\frac{D \times (t + t' - 64)}{900} = \frac{1362 \times 56{\cdot}7}{900} = \quad 85{\cdot}8$$

Second approximate difference, D′ = 1,447·5
Table III gives for D′ = 1448 and lat. 36°... 1·2
" IV " for D′ = 1448 .... 3·8
" V " for D′ = 1448 and $h = 25$ ... 0·7
Highest peak above Mountain House, or Z = 1,453·2
Mountain House above the sea............. 5,248·4
Black Mountain, highest peak above the sea,
or altitude............................ 6,701·6 Eng. ft.

I.

TABLES

FOR COMPUTING THE DIFFERENCE IN THE HEIGHT OF TWO PLACES FROM BAROMETRICAL OBSERVATIONS.

I. $D = 60158{\cdot}58 \times \log. H$ or h. Argument, the observed height of the barometer at either station.

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH.										Thousandths of an inch.		Barometer in Eng. inch.
	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09			
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
12·0	4763·4	4785·2	4806·9	4828·7	4850·4	4872·1	4893·7	4915·4	4937·0	4958·6			12·0
12·1	4980·2	5001·8	5023·4	5044·9	5066·4	5087·9	5109·4	5130·9	5152·4	5173·8			12·1
12·2	5195·2	5216·6	5238·0	5259·4	5280·7	5302·1	5323·4	5344·7	5367·0	5387·2			12·2
12·3	5408·5	5429·8	5452·0	5472·2	5493·4	5514·5	5535·7	5556·8	5578·9	5599·0	1	2·1	12·3
12·4	5620·1	5641·2	5662·2	5683·2	5704·3	5725·3	5746·2	5767·2	5788·1	5809·0	2	4·2	12·4
12·5	5829·9	5850·8	5871·7	5892·6	5913·4	5934·2	5955·0	5975·8	5996·6	6017·4	3	6·2	12·5
12·6	6038·1	6058·8	6079·6	6100·2	6120·9	6141·6	6162·2	6182·8	6203·5	6224·0	4	8·3	12·6
12·7	6244·6	6265·2	6285·8	6306·3	6326·8	6347·3	6367·8	6388·3	6408·8	6429·2	5	10·4	12·7
12·8	6449·6	6470·0	6490·4	6510·8	6531·1	6551·5	6571·8	6592·1	6612·4	6632·7	6	12·5	12·8
12·9	6652·9	6673·2	6693·4	6713·6	6733·8	6754·0	6774·1	6794·3	6814·4	6834·5	7	14·6	12·9
13·0	6854·7	6874·7	6894·8	6914·9	6934·9	6955·0	6975·0	6995·0	7014·9	7034·9	8	16·6	13·0
13·1	7054·9	7074·8	7094·7	7114·6	7134·5	7154·4	7174·3	7194·1	7213·9	7233·8	9	18·7	13·1
13·2	7253·6	7273·3	7293·1	7312·9	7332·6	7352·3	7372·1	7391·8	7411·4	7431·1			13·2
13·3	7450·8	7470·4	7490·0	7509·6	7529·2	7548·8	7568·4	7587·9	7607·4	7627·0			13·3
13·4	7646·5	7666·0	7685·4	7704·9	7724·4	7743·8	7763·2	7782·6	7802·0	7821·4			13·4

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
13·5	7840·8	7860·1	7879·4	7898·7	7918·0	7937·3	7956·6	7975·8	7995·1	8014·3			13·5
13·6	8033·6	8052·8	8071·9	8091·1	8110·3	8129·4	8148·6	8167·7	8186·8	8205·9			13·6
13·7	8225·0	8244·0	8263·1	8282·1	8301·1	8320·1	8339·1	8358·1	8377·1	8396·0	1	1·9	13·7
13·8	8415·0	8433·9	8452·8	8471·7	8490·6	8509·4	8528·3	8547·1	8565·9	8584·8	2	3·8	13·8
13·9	8603·6	8622·3	8641·1	8659·9	8678·6	8697·4	8716·1	8734·8	8753·5	8772·2	3	5·6	13·9
14·0	8790·8	8809·5	8828·2	8846·8	8865·4	8884·0	8902·6	8921·2	8939·7	8958·3	4	7·5	14·0
14·1	8976·8	8995·4	9013·9	9032·4	9050·8	9069·3	9087·8	9106·2	9124·6	9143·0	5	9·4	14·1
14·2	9161·4	9179·8	9198·2	9216·6	9234·9	9253·3	9271·6	9289·9	9308·2	9326·5	6	11·3	14·2
14·3	9344·7	9363·0	9381·3	9399·5	9417·7	9436·0	9454·2	9472·3	9490·5	9508·7	7	13·2	14·3
14·4	9526·8	9545·0	9563·1	9581·2	9599·3	9617·4	9635·5	9653·5	9671·6	9689·6	8	15·0	14·4
14·5	9707·6	9725·7	9743·7	9761·7	9779·6	9797·6	9815·6	9833·5	9851·4	9869·3	9	17·0	14·5
14·6	9887·2	9905·1	9923·0	9940·9	9958·7	9976·5	9994·4	10012·2	10030·0	10047·8			14·6
14·7	10065·5	10083·3	10101·1	10118·8	10136·6	10154·3	10172·0	10189·7	10207·4	10225·1			14·7
14·8	10242·7	10260·4	10278·0	10295·7	10313·3	10330·9	10348·5	10366·1	10383·6	10401·2	1	1·7	14·8
14·9	10418·7	10436·3	10453·8	10471·3	10488·8	10506·3	10523·7	10541·2	10558·6	10576·0	2	3·4	14·9
15·0	10593·4	10610·8	10628·2	10645·6	10662·9	10680·3	10697·6	10715·0	10732·3	10749·6	3	5·1	15·0
15·1	10766·9	10784·1	10801·5	10818·7	10836·0	10853·2	10870·5	10887·7	10904·9	10922·1	4	6·8	15·1
15·2	10939·3	10956·5	10973·6	10990·8	11008·0	11025·1	11042·2	11059·3	11076·4	11093·5	5	8·5	15·2
15·3	11110·6	11127·7	11144·7	11161·8	11178·8	11195·8	11212·8	11229·8	11246·8	11263·8	6	10·2	15·3
15·4	11280·8	11297·8	11314·7	11331·6	11348·6	11365·5	11382·4	11399·3	11416·2	11433·0	7	11·9	15·4
15·5	11449·9	11466·7	11483·6	11500·4	11517·2	11534·0	11550·8	11567·6	11584·4	11601·1	8	13·6	15·5
15·6	11617·9	11634·6	11651·4	11668·1	11684·8	11701·5	11718·2	11734·9	11751·6	11768·2	9	15·3	15·6
15·7	11784·9	11801·5	11818·2	11834·8	11851·4	11868·0	11884·6	11901·1	11917·7	11934·3			15·7
15·8	11950·8	11967·3	11983·8	12000·4	12016·9	12033·3	12049·8	12066·3	12082·7	12099·2			15·8
15·9	12115·6	12132·0	12148·4	12164·8	12181·2	12197·6	12214·0	12230·4	12246·7	12263·1			15·9

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
16·0	12279·6	12295·9	12312·2	12328·5	12344·8	12361·1	12377·4	12393·6	12409·9	12426·1			16·0
16·1	12442·4	12458·6	12474·8	12491·0	12507·2	12523·4	12539·6	12555·7	12571·9	12588·0			16·1
16·2	12604·2	12620·3	12636·4	12652·5	12668·6	12684·7	12700·8	12716·8	12732·9	12748·9	1	1·6	16·2
16·3	12765·0	12781·0	12797·0	12813·0	12829·0	12845·0	12861·0	12876·9	12892·9	12908·8	2	3·1	16·3
16·4	12924·8	12940·7	12956·6	12972·5	12988·4	13004·3	13020·2	13036·0	13051·9	13067·7	3	4·7	16·4
16·5	13083·6	13099·4	13115·2	13131·0	13146·8	13162·6	13178·4	13194·2	13210·0	13225·7	4	6·3	16·5
16·6	13241·5	13257·2	13272·9	13288·6	13304·3	13320·0	13335·7	13351·4	13367·1	13382·7	5	7·8	16·6
16·7	13398·4	13414·0	13429·6	13445·2	13460·8	13476·4	13492·0	13507·6	13523·2	13538·7	6	9·4	16·7
16·8	13554·3	13569·8	13585·4	13600·9	13616·4	13631·9	13647·4	13662·9	13678·4	13693·9	7	11·0	16·8
16·9	13709·4	13724·8	13740·3	13755·7	13771·1	13786·5	13801·9	13817·3	13832·7	13848·1	8	12·5	16·9
17·0	13863·5	13878·8	13894·2	13909·6	13924·9	13940·2	13955·6	13970·9	13986·2	14001·5	9	14·1	17·0
17·1	14016·8	14032·0	14047·3	14062·6	14077·8	14093·0	14108·3	14123·5	14138·7	14153·9			17·1
17·2	14169·1	14184·3	14199·4	14214·6	14229·8	14244·9	14260·1	14275·2	14290·3	14305·5			17·2
17·3	14320·6	14335·7	14350·8	14365·8	14380·9	14396·0	14411·0	14426·1	14441·1	14456·2			17·3
17·4	14471·2	14486·2	14501·2	14516·2	14531·2	14546·1	14561·1	14576·1	14591·0	14605·9	1	1·5	17·4
17·5	14620·9	14635·8	14650·7	14665·6	14680·5	14695·4	14710·3	14725·2	14740·1	14754·9	2	2·9	17·5
17·6	14769·8	14784·6	14799·4	14814·3	14829·1	14843·9	14858·7	14873·5	14888·2	14903·0	3	4·4	17·6
17·7	14917·8	14932·5	14947·3	14962·0	14976·8	14991·5	15006·2	15020·9	15035·6	15050·3	4	5·8	17·7
17·8	15065·0	15079·6	15094·3	15109·0	15123·6	15138·2	15152·9	15167·5	15182·1	15196·7	5	7·3	17·8
17·9	15211·3	15225·9	15240·5	15255·0	15269·6	15284·2	15298·7	15313·3	15327·8	15342·4	6	8·8	17·9
18·0	15356·8	15371·3	15385·8	15400·3	15414·8	15429·3	15443·7	15458·2	15472·7	15487·1	7	10·2	18·0
18·1	15501·5	15516·0	15530·4	15544·8	15559·2	15573·6	15588·0	15602·4	15616·8	15631·2	8	11·7	18·1
18·2	15645·5	15659·9	15674·2	15688·5	15702·9	15717·2	15731·5	15745·8	15760·1	15774·4	9	13·1	18·2
18·3	15788·6	15802·9	15817·2	15831·4	15845·7	15859·9	15874·2	15888·4	15902·6	15916·8			18·3
18·4	15931·0	15945·2	15959·4	15973·6	15987·8	16001·9	16016·1	16030·2	16044·4	16058·5			18·4

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
18·5	16072·6	16086·8	16100·9	16115·0	16129·1	16143·2	16157·3	16171·3	16185·4	16199·5			18·5
18·6	16213·5	16227·6	16241·6	16255·6	16269·7	16283·7	16297·7	16311·7	16325·7	16339·6			18·6
18·7	16353·5	16367·5	16381·5	16395·4	16409·4	16423·3	16437·2	16451·2	16465·1	16479·0	1	1·4	18·7
18·8	16492·9	16506·8	16520·7	16534·6	16548·5	16562·3	16576·2	16590·0	16603·9	16617·8	2	2·7	18·8
18·9	16631·5	16645·4	16659·2	16673·0	16686·8	16700·6	16714·4	16728·1	16741·9	16755·7	3	4·1	18·9
19·0	16769·4	16783·2	16796·9	16810·6	16824·3	16838·1	16851·8	16865·5	16879·2	16892·8	4	5·4	19·0
19·1	16906·5	16920·2	16933·9	16947·5	16961·2	16974·9	16988·5	17002·1	17015·8	17029·4	5	6·8	19·1
19·2	17043·0	17056·6	17070·2	17083·8	17097·4	17110·9	17124·5	17138·1	17151·6	17165·2	6	8·1	19·2
19·3	17178·7	17192·2	17205·8	17219·3	17232·8	17246·3	17259·8	17273·3	17286·8	17300·3	7	9·5	19·3
19·4	17313·7	17327·2	17340·6	17354·1	17367·5	17380·9	17394·4	17407·8	17421·2	17434·6	8	10·9	19·4
19·5	17448·0	17461·4	17474·8	17488·2	17501·6	17515·0	17528·3	17541·7	17555·0	17568·4	9	12·2	19·5
19·6	17581·7	17595·0	17608·3	17621·7	17635·0	17648·2	17661·5	17674·8	17688·1	17701·4			19·6
19·7	17714·6	17727·9	17741·1	17754·4	17767·6	17780·8	17794·1	17807·3	17820·5	17833·7			19·7
19·8	17846·9	17860·1	17873·3	17886·5	17899·6	17912·8	17926·0	17939·1	17952·2	17965·4			19·8
19·9	17978·5	17991·6	18004·8	18017·9	18031·0	18044·1	18057·2	18070·3	18083·4	18096·4	1	1·3	19·9
20·0	18109·5	18122·6	18135·6	18148·7	18161·7	18174·8	18187·8	18200·8	18213·8	18226·8	2	2·6	20·0
20·1	18239·8	18252·8	18265·8	18278·8	18291·8	18304·8	18317·7	18330·7	18343·6	18356·6	3	3·9	20·1
20·2	18369·5	18382·5	18395·4	18408·3	18421·2	18434·1	18447·0	18459·9	18472·8	18485·7	4	5·1	20·2
20·3	18498·5	18511·4	18524·3	18537·1	18550·0	18562·8	18575·7	18588·5	18601·3	18614·1	5	6·4	20·3
20·4	18626·9	18639·7	18652·5	18665·3	18678·1	18690·9	18703·6	18716·4	18729·1	18741·9	6	7·7	20·4
20·5	18754·6	18767·4	18780·1	18792·9	18805·6	18818·3	18831·0	18843·7	18856·4	18869·1	7	9·0	20·5
20·6	18881·8	18894·5	18907·2	18919·9	18932·5	18945·2	18957·8	18970·5	18983·1	18995·7	8	10·3	20·6
20·7	19008·3	19021·0	19033·6	19046·2	19058·8	19071·4	19083·9	19096·5	19109·1	19121·7	9	11·6	20·7
20·8	19134·2	19146·8	19159·3	19171·9	19184·4	19196·9	19209·5	19222·0	19234·5	19247·0			20·8
20·9	19259·5	19272·0	19284·5	19297·1	19309·5	19322·0	19334·4	19346·9	19359·3	19371·8			20·9

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
21·0	19384·3	19396·7	19409·1	19421·5	19434·0	19446·4	19458·8	19471·2	19483·6	19496·0			21·0
21·1	19508·4	19520·8	19533·1	19545·5	19557·9	19570·2	19582·6	19594·9	19607·3	19619·6	1	1·2	21·1
21·2	19632·0	19644·3	19656·6	19668·9	19681·2	19693·5	19705·8	19718·0	19730·3	19742·6	2	2·4	21·2
21·3	19754·9	19767·1	19779·4	19791·6	19803·9	19816·1	19828·4	19840·6	19852·8	19865·0	3	3·6	21·3
21·4	19877·3	19889·5	19901·7	19913·9	19926·0	19938·2	19950·4	19962·6	19974·7	19986·9	4	4·8	21·4
21·5	19999·1	20011·2	20023·3	20035·5	20047·6	20059·7	20071·8	20083·9	20096·1	20108·2	5	6·0	21·5
21·6	20120·3	20132·3	20144·4	20156·5	20168·6	20180·7	20192·7	20204·8	20216·9	20228·9	6	7·2	21·6
21·7	20241·0	20253·0	20265·0	20277·0	20289·1	20301·1	20313·1	20325·1	20337·1	20349·1	7	8·4	21·7
21·8	20361·1	20373·0	20385·0	20397·0	20409·0	20420·9	20432·9	20444·8	20456·8	20468·7	8	9·7	21·8
21·9	20480·7	20492·6	20504·5	20516·4	20528·3	20540·2	20552·1	20564·0	20575·9	20587·8	9	10·9	21·9
22·0	20599·7	20611·5	20623·4	20635·3	20647·1	20659·0	20670·8	20682·7	20694·5	20706·3			22·0
22·1	20718·2	20732·0	20741·8	20753·6	20765·4	20777·2	20789·0	20801·8	20812·6	20824·4			22·1
22·2	20836·2	20847·9	20859·7	20871·4	20883·2	20894·9	20906·7	20918·4	20930·1	20941·9			22·2
22·3	20953·6	20965·3	20977·0	20988·7	21000·4	21012·1	21023·8	21035·4	21047·1	21058·8	1	1·1	22·3
22·4	21070·5	21082·1	21093·8	21105·4	21117·1	21128·7	21140·4	21152·0	21163·6	21175·3	2	2·3	22·4
22·5	21186·9	21198·5	21210·1	21221·6	21233·2	21244·8	21256·4	21268·0	21279·5	21291·1	3	3·4	22·5
22·6	21302·6	21314·2	21325·8	21337·3	21348·9	21360·4	21371·9	21383·5	21395·0	21406·5	4	4·6	22·6
22·7	21418·1	21429·6	21441·1	21452·5	21464·0	21475.5	21487·0	21498·5	21509·9	21521·4	5	5·7	22·7
22·8	21532·9	21544·3	21555·8	21567·2	21578·7	21590·1	21601·6	21613·0	21624·4	21635·8	6	6·8	22·8
22·9	21647·3	21658·7	21670·1	21681·4	21692·8	21704·2	21715·6	21727·0	21738·3	21749·7	7	8·0	22·9
23·0	21761·0	21772·4	21783·7	21795·1	21806·4	21817·7	21829·1	21840·4	21851·7	21863·0	8	9·1	23·0
23·1	21874·3	21885·6	21897·0	21908·3	21919·6	21930·8	21942·1	21953·4	21964·7	21976·0	9	10·2	23·1
23·2	21987·2	21998·5	22009·8	22021·0	22032·3	22043·5	22054·7	22066·0	22077·2	22088·4			23·2
23·3	22099·6	22110·8	22122·1	22133·3	22144·5	22155·6	22166·8	22178·0	22189·2	22200·4			23·3
23·4	22211·5	22222·7	22233·9	22245·0	22256·0	22267·3	22278·4	22289·6	22300·7	22311·8			23·4

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
23·5	22322·9	22334·0	22345·2	22356·3	22367·4	22378·4	22389·5	22400·6	22411·7	22422·8			23·5
23·6	22433·8	22444·9	22456·0	22467·0	22478·1	22489·1	22500·2	22511·2	22522·3	22533·3			23·6
23·7	22544·3	22555·4	22566·4	22577·4	22588·4	22599·4	22610·4	22621·4	22632·4	22643·4			23·7
23·8	22654·3	22665·3	22676·3	22687·2	22698·2	22709·1	22720·1	22731·0	22742·0	22752·9	1	1·1	23·8
23·9	22763·8	22774·8	22785·7	22796·6	22807·5	22818·4	22829·4	22840·3	22851·2	22862·0	2	2·2	23·9
24·0	22873·0	22883·9	22894·7	22905·6	22916·5	22927·4	22938·2	22949·1	22960·0	22970·8	3	3·2	24·0
24·1	22981·7	22992·5	23003·3	23014·2	23025·0	23035·8	23046·6	23057·5	23068·3	23079·1	4	4·3	24·1
24·2	23089·9	23100·7	23111·4	23122·2	23133·0	23143·8	23154·5	23165·3	23176·1	23186·8	5	5·4	24·2
24·3	23197·6	23208·3	23219·1	23229·8	23240·5	23251·3	23262·0	23272·7	23283·4	23294·2	6	6·5	24·3
24·4	23304·9	23315·6	23326·3	23337·0	23347·6	23358·3	23369·0	23379·7	23390·3	23401·0	7	7·5	24·4
24·5	23411·7	23422·3	23433·0	23443·7	23454·3	23464·9	23475·6	23486·2	23496·8	23507·4	8	8·6	24·5
24·6	23518·1	23528·7	23539·3	23549·9	23560·5	23571·1	23581·7	23592·3	23602·9	23613·5	9	9·7	24·6
24·7	23624·1	23634·6	23645·2	23655·8	23666·3	23676·9	23687·5	23698·0	23708·6	23719·1			24·7
24·8	23729·7	23740·2	23750·7	23761·2	23771·7	23782·3	23792·8	23803·3	23813·8	23824·3			24·8
24·9	23834·8	23845·3	23855·7	23866·2	23876·7	23887·2	23897·7	23908·2	23918·6	23929·1	1	1·0	24·9
25·0	23939·5	23949·9	23960·4	23970·8	23981·3	23991·7	24002·1	24012·5	24023·0	24033·4	2	2·1	25·0
25·1	24043·8	24054·2	24064·6	24075·0	24085·4	24095·7	24106·1	24116·5	24126·9	24137·2	3	3·1	25·1
25·2	24147·6	24158·0	24168·3	24178·7	24189·0	24199·4	24209·7	24220·1	24230·4	24240·8	4	4·1	25·2
25·3	24251·1	24261·4	24271·8	24282·1	24292·4	24302·7	24313·0	24323·3	24333·6	24343·9	5	5·1	25·3
25·4	24354·2	24364·5	24374·7	24385·0	24395·3	24405·5	24415·8	24426·1	24436·3	24446·6	6	6·2	25·4
25·5	24456·8	24467·0	24477·3	24487·5	24497·8	24508·0	24518·2	24528·4	24538·7	24548·9	7	7·2	25·5
25·6	24559·1	24569·3	24579·5	24589·7	24599·9	24610·0	24620·2	24630·4	24640·6	24650·7	8	8·2	25·6
25·7	24660·9	24671·1	24681·2	24691·4	24701·5	24711·7	24721·8	24732·0	24742·1	24752·3	9	9·2	25·7
25·8	24762·4	24772·5	24782·6	24792·8	24802·9	24813·0	24823·1	24833·2	24843·3	24853·4			25·8
25·9	24863·5	24873·6	24883·7	24893·7	24903·8	24913·9	24924·0	24934·0	24944·1	24954·1			25·9

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
26·0	24964·2	24974·2	24984·3	24994·3	25004·4	25014·4	25024·4	25034·4	25044·5	25054·5			26·0
26·1	25064·5	25074·5	25084·5	25094·5	25104·5	25114·5	25124·5	25134·5	25144·4	25154·4			26·1
26·2	25164·4	25174·4	25184·3	25194·3	25204·2	25214·2	25224·1	25234·1	25244·0	25254·0	1	1·0	26·2
26·3	25263·9	25273·8	25283·8	25293·7	25303·6	25313·5	25323·4	25333·3	25343·2	25353·1	2	2·0	26·3
26·4	25363·0	25372·9	25382·8	25392·7	25402·6	25412·4	25422·3	25432·2	25442·1	25451·9	3	2·9	26·4
26·5	25461·8	25471·7	25481·5	25491·4	25501·2	25511·0	25520·9	25530·7	25540·5	25550·4	4	3·9	26·5
26·6	25560·2	25570·0	25579·8	25589·7	25599·5	25609·3	25619·1	25628·9	25638·7	25648·5	5	4·9	26·6
26·7	25658·3	25668·1	25677·8	25687·6	25697·4	25707·1	25716·9	25726·7	25736·4	25746·2	6	5·9	26·7
26·8	25755·9	25765·6	25775·4	25785·1	25794·8	25804·6	25814·3	25824·0	25833·8	25843·5	7	6·9	26·8
26·9	25853·2	25862·9	25872 6	25882·3	25892·0	25901·7	25911·4	25921·1	25930·8	25940·5	8	7·8	26·9
27·0	25950·2	25959·9	25969·6	25979·2	25988·9	25998·6	26008·2	26017·9	26027·5	26037·2	9	8·8	27·0
27·1	26046·8	26056·5	26066·1	26075·7	26085·3	26095·0	26104·6	26114·2	26123·8	26133·4			27·1
27·2	26143·0	26152·6	26162·2	26171·8	26181·4	26191·0	26200·6	26210·2	26219·8	26229·3			27·2
27·3	26238·9	26248·5	26258·0	26267·6	26277·2	26286·7	26296·3	26305·8	26315 3	26324·9			27·3
27·4	26334·4	26344·0	26353·5	26363·0	26372·5	26382·1	26391·6	26401·1	26410·6	26420·1	1	0·9	27·4
27·5	26429·6	26439·1	26448·6	26458·1	26467·6	26477·1	26486·5	26496·0	26505·5	26514·9	2	1·9	27·5
27·6	26524·4	26533·9	26543·3	26552·8	26562·3	26571·7	26581·2	26590·6	26600·0	26609·5	3	2·8	27·6
27·7	26618·9	26628·4	26637·8	26647·2	26656·7	26666·1	26675·5	26684·9	26694·3	26703·7	4	3·7	27·7
27·8	26713·1	26722·5	26731·9	26741·3	26750·7	26760·1	26769·5	26778·8	26788·2	26797·6	5	4·7	27·8
27·9	26806·9	26816·3	26825·6	26835·0	26844·3	26853·7	26863·0	26872·3	26881·7	26891·0	6	5·6	27·9
28·0	26900·4	26909·7	26919·0	26928·4	26937·7	26947·0	26956·3	26965·6	26975·0	26984·3	7	6·5	28·0
28·1	26993·6	27002·9	27012·2	27021·5	27030·7	27040·0	27049·3	27058·6	27067·8	27077·1	8	7·5	28·1
28·2	27086·4	27095·6	27104·9	27114·2	27123·4	27132·7	27141·9	27151·2	27160·4	27169·6	9	8·4	28·2
28·3	27178·9	27188·1	27197·3	27206·5	27215·7	27225·0	27234·2	27243·4	27252·6	27261·8			28·3
28·4	27271·0	27280·2	27289·4	27298·6	27307·8	27317·0	27326·2	27335·3	27344·5	27353·7			28·4

Barometer in Eng. inch.	HUNDREDTHS OF AN INCH. ·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Thousandths of an inch.		Barometer in Eng. inch.
	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.	Eng. feet.		Feet.	
28·5	27362·9	27372·0	27381·2	27390·4	27399·5	27408·7	27417·8	27427·0	27436·1	27445·2			28·5
28·6	27454·4	27463·5	27472·6	27481·8	27490·9	27500·0	27509·1	27518·2	27527·4	27536·5			28·6
28·7	27545·6	27554·7	27563·8	27572·9	27582·0	27591·1	27600·2	27609·3	27618·3	27627·4	1	0·9	28·7
28·8	27636·5	27645·5	27654·6	27663·7	27672·7	27681·8	27690·8	27699·9	27708·9	27717·9	2	1·8	28·8
28·9	27727·0	27736·0	27745·1	27754·1	27763·1	27772·2	27781·2	27790·2	27799·2	27808·3	3	2·7	28·9
29·0	27817·2	27826·2	27835·2	27844·2	27853·2	27862·2	27871·2	27880·2	27889·1	27898·1	4	3·6	29·0
29·1	27907·1	27916·1	27925·0	27934·0	27943·0	27951·9	27960·9	27969·8	27978·8	27987·7	5	4·5	29·1
29·2	27996·7	28005·6	28014·6	28023·5	28032·4	28041·4	28050·3	28059·2	28068·2	28077·1	6	5·4	29·2
29·3	28086·0	28094·9	28103·8	28112·8	28121·7	28130·6	28139·5	28148·4	28157·3	28166·2	7	6·3	29·3
29·4	28175·1	28184·0	28192·9	28201·7	28210·6	28219·5	28228·4	28237·2	28246·1	28254·9	8	7·2	29·4
29·5	28263·8	28272·6	28281·5	28290·3	28299·2	28308·0	28316·9	28325·7	28334·5	28343·4	9	8·1	29·5
29·6	28352·2	28361·0	28369·8	28378·7	28387·5	28396·3	28405·1	28413·9	28422·7	28431·5			29·6
29·7	28440·3	28449·1	28457·9	28466·7	28475·4	28484·2	28493·0	28501·8	28510·6	28519·3			29·7
29·8	28528·1	28536·9	28545·6	28554·4	28563·2	28571·9	28580·7	28589·4	28598·2	28606·9			29·8
29·9	28615·7	28624·4	28633·2	28641·9	28650·6	28659·3	28668·1	28676·8	28685·5	28694·2	1	8·6	29·9
30·0	28702·9	28711·6	28720·3	28729·0	28737·7	28746·4	28755·1	28763·8	28772·5	28781·1	2	1·7	30·0
30·1	28789·8	28798·5	28807·2	28815·9	28824·5	28833·2	28841·9	28850·5	28859·2	28867·9	3	2·6	30·1
30·2	28876·5	28885·2	28893·8	28902·5	28911·1	28919·8	28928·4	28937·0	28945·7	28954·3	4	3·4	30·2
30·3	28962·9	28971·5	28980·1	28988·8	28997·4	29006·0	29014·6	29023·2	29031·7	29040·3	5	4·3	30·3
30·4	29048·9	29057·5	29066·1	29074·7	29083·3	29091·8	29100·4	29109·0	29117·6	29126·2	6	5·2	30·4
30·5	29134·7	29143·3	29151·9	29160·4	29169·0	29177·6	29186·1	29194·7	29203·2	29211·8	7	6·0	30·5
30·6	29220·3	29228·9	29237·4	29245·9	29254·4	29262·9	29271·5	29280·0	29288·5	29297·0	8	6·9	30·6
30·7	29305·5	29314·0	29322·5	29331·1	29339·6	29348·1	29356·6	29365·1	29373·5	29382·0	9	7·7	30·7
30·8	29390·5	29399·0	29407·5	29416·0	29424·4	29432·9	29441·4	29449·8	29458·3	29466·8			30·8
30·9	29475·2	29483·7	29492·1	29500·6	29509·0	29517·5	29525·9	29534·3	29542·8	29551·2			30·9

II. Correction for $\tau-\tau'$, or Difference of the Temperature of the Barometers at the Two Stations.

This correction is *negative* when the attached thermometer at the upper station is lowest; *positive*, when the attached thermometer at the upper station is highest.

$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.	$\tau-\tau'$ Fahrenheit.	Correction in Eng. feet.
°		°		°		°		°		°		°		°		°		°	
1·0	2·3	11·0	25·8	21·0	49·2	31·0	72·6	41·0	96·0	51·0	119·5	61·0	142·9	71·0	166·3	81·0	189·7	91·0	213·2
1·5	3·5	11·5	26·9	21·5	50·4	31·5	73·8	41·5	97·2	51·5	120·6	61·5	144·1	71·5	167·5	81·5	190·9	91·5	214·3
2·0	4·7	12·0	28·1	22·0	51·5	32·0	75·0	42·0	98·4	52·0	121·8	62·0	145·2	72·0	168·7	82·0	192·1	92·0	215·5
2·5	5·9	12·5	29·3	22·5	52·7	32·5	76·1	42·5	99·6	52·5	123·0	62·5	146·4	72·5	169·8	82·5	193·3	92·5	216·7
3·0	7·0	13·0	30·5	23·0	53·9	33·0	77·3	43·0	100·7	53·0	124·2	63·0	147·6	73·0	171·0	83·0	194·4	93·0	217·9
3·5	8·2	13·5	31·6	23·5	55·1	33·5	78·5	43·5	101·9	53·5	125·3	63·5	148·8	73·5	172·2	83·5	195·6	93·5	219·0
4·0	9·4	14·0	32·8	24·0	56·2	34·0	79·6	44·0	103·1	54·0	126·5	64·0	149·9	74·0	173·4	84·0	196·8	94·0	220·2
4·5	10·5	14·5	34·0	24·5	57·4	34·5	80·8	44·5	104·2	54·5	127·7	64·5	151·1	74·5	174·5	84·5	197·9	94·5	221·4
5·0	11·7	15·0	35·1	25·0	58·6	35·0	82·0	45·0	105·4	55·0	128·8	65·0	152·3	75·0	175·7	85·0	199·1	95·0	222·5
5·5	12·9	15·5	36·3	25·5	59·7	35·5	83·2	45·5	106·6	55·5	130·0	65·5	153·4	75·5	176·9	85·5	200·3	95·5	223·7
6·0	14·1	16·0	37·5	26·0	60·9	36·0	84·3	46·0	107·8	56·0	131·2	66·0	154·6	76·0	178·0	86·0	201·5	96·0	224·9
6·5	15·2	16·5	38·7	26·5	62·1	36·5	85·5	46·5	108·9	56·5	132·4	66·5	155·8	76·5	179·2	86·5	202·6	96·5	226·1
7·0	16·4	17·0	39·8	27·0	63·2	37·0	86·7	47·0	110·1	57·0	133·5	67·0	157·0	77·0	180·4	87·0	203·8	97·0	227·2
7·5	17·6	17·5	41·0	27·5	64·4	37·5	87·8	47·5	111·3	57·5	134·7	67·5	158·1	77·5	181·6	87·5	205·0	97·5	228·4
8·0	18·7	18·0	42·2	28·0	65·6	38·0	89·0	48·0	112·4	58·0	135·9	68·0	159·3	78·0	182·7	88·0	206·1	98·0	229·6
8·5	19·9	18·5	43·3	28·5	66·8	38·5	90·2	48·5	113·6	58·5	137·0	68·5	160·5	78·5	183·9	88·5	207·3	98·5	230·7
9·0	21·1	19·0	44·5	29·0	67·9	39·0	91·4	49·0	114·8	59·0	138·2	69·0	161·6	79·0	185·1	89·0	208·5	99·0	231·9
9·5	22·3	19·5	45·7	29·5	69·1	39·5	92·5	49·5	116·0	59·5	139·4	69·5	162·8	79·5	186·2	89·5	209·7	99·5	233·1
10·0	23·4	20·0	46·9	30·0	70·3	40·0	93·7	50·0	117·1	60·0	140·6	70·0	164·0	80·0	187·4	90·0	210·8	100·0	234·3
10·5	24·6	20·5	48·0	30·5	71·4	40·5	94·9	50·5	118·3	60·5	141·7	70·5	165·2	80·5	188·6	90·5	212·0	100·5	235·4

III. CORRECTION FOR THE DIFFERENCE OF GRAVITY IN VARIOUS LATITUDES.

Correction *positive* from latitude 0° to 45°; *negative* from 45° to 90°.

Approximate difference of level.	LATITUDE. 0° 90°	2° 88°	4° 86°	6° 84°	8° 82°	10° 80°	12° 78°	14° 76°	16° 74°	18° 72°	20° 70°	22° 68°	24° 66°	26° 64°	28° 62°	30° 60°	32° 58°	34° 56°	36° 54°	38° 52°	40° 50°	42° 48°	44° 46°	45°
Eng. ft.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
1000	2·6	2·6	2·6	2·5	2·5	2·4	2·4	2·3	2·2	2·1	2·0	1·9	1·7	1·6	1·5	1·3	1·1	1·0	0·8	0·6	0·5	0·3	0·1	0
2000	5·2	5·2	5·1	5·1	5·0	4·9	4·7	4·6	4·4	4·2	4·0	3·7	3·5	3·2	2·9	2·6	2·3	1·9	1·6	1·3	0·9	0·5	0·2	0
3000	7·8	7·8	7·7	7·6	7·5	7·3	7·1	6·9	6·6	6·3	6·0	5·6	5·2	4·8	4·4	3·9	3·4	2·9	2·4	1·9	1·4	0·8	0·3	0
4000	10·4	10·4	10·3	10·2	10·0	9·8	9·5	9·2	8·8	8·4	8·0	7·5	7·0	6·4	5·8	5·2	4·6	3·9	3·2	2·5	1·8	1·1	0·4	0
5000	13·0	13·0	12·9	12·7	12·5	12·2	11·9	11·5	11·0	10·5	10·0	9·4	8·7	8·0	7·3	6·5	5·7	4·9	4·0	3·1	2·3	1·4	0·5	0
6000	15·6	15·6	15·4	15·3	15·0	14·7	14·3	13·8	13·2	12·6	11·9	11·2	10·4	9·6	8·7	7·8	6·8	5·8	4·8	3·8	2·7	1·6	0·5	0
7000	18·2	18·2	18·0	17·8	17·5	17·1	16·6	16·1	15·4	14·7	13·9	13·1	12·2	11·2	10·2	9·1	8·0	6·8	5·6	4·4	3·2	1·9	0·6	0
8000	20·8	20·7	20·6	20·3	20·0	19·5	19·0	18·4	17·6	16·8	15·9	15·0	13·9	12·8	11·6	10·4	9·1	7·8	6·4	5·0	3·6	2·2	0·7	0
9000	23·4	23·3	23·2	22·9	22·5	22·0	21·4	20·7	19·8	18·9	17·9	16·8	15·7	14·4	13·1	11·7	10·3	8·8	7·2	5·7	4·1	2·4	0·8	0
10000	26·0	25·9	25·7	25·4	25·0	24·4	23·8	23·0	22·0	21·0	19·9	18·7	17·4	16·0	14·5	13·0	11·4	9·7	8·0	6·3	4·5	2·7	0·9	0
11000	28·6	28·5	28·3	28·0	27·5	26·9	26·1	25·3	24·3	23·1	21·9	20·6	19·1	17·6	16·0	14·3	12·5	10·7	8·8	6·9	5·0	3·0	1·0	0
12000	31·2	31·1	30·9	30·5	30·0	29·3	28·5	27·5	26·5	25·2	23·9	22·4	20·9	19·2	17·4	15·6	13·7	11·7	9·6	7·5	5·4	3·3	1·1	0
13000	33·8	33·7	33·5	33·1	32·5	31·8	30·9	29·8	28·7	27·3	25·9	24·3	22·6	20·8	18·9	16·9	14·8	12·7	10·4	8·2	5·9	3·5	1·2	0
14000	36·4	36·3	36·0	35·6	35·0	34·2	33·3	32·1	30·9	29·4	27·9	26·2	24·4	22·4	20·4	18·2	16·0	13·6	11·2	8·8	6·3	3·8	1·3	0
15000	39·0	38·9	38·6	38·1	37·5	36·6	35·6	34·4	33·1	31·6	29·9	28·1	26·1	24·0	21·8	19·5	17·1	14·6	12·1	9·4	6·8	4·1	1·4	0
16000	41·6	41·5	41·2	40·7	40·0	39·1	38·0	36·7	35·3	33·7	31·9	29·9	27·8	25·6	23·3	20·8	18·2	15·6	12·9	10·1	7·2	4·3	1·5	0
17000	44·2	44·1	43·8	43·2	42·5	41·5	40·4	39·0	37·5	35·8	33·9	31·8	29·6	27·2	24·7	22·1	19·4	16·6	13·7	10·7	7·7	4·6	1·5	0
18000	46·8	46·7	46·3	45·8	45·0	44·0	42·8	41·3	39·7	37·9	35·8	33·7	31·3	28·8	26·2	23·4	20·5	17·5	14·5	11·3	8·1	4·9	1·6	0
19000	49·4	49·3	48·9	48·3	47·5	46·4	45·1	43·6	41·9	40·0	37·8	35·5	33·1	30·4	27·6	24·7	21·7	18·5	15·3	12·0	8·6	5·2	1·7	0
20000	52·0	51·9	51·5	50·9	50·0	48·9	47·5	45·9	44·1	42·1	39·8	37·4	34·8	32·0	29·1	26·0	22·8	19·5	16·1	12·6	9·0	5·4	1·8	0
21000	54·6	54·5	54·1	53·4	52·5	51·3	49·9	48·2	46·3	44·2	41·8	39·3	36·5	33·6	30·5	27·3	23·9	20·5	16·9	13·2	9·5	5·7	1·9	0
22000	57·2	57·1	56·6	55·9	55·0	53·7	52·3	50·5	48·5	46·3	43·8	41·1	38·3	35·2	32·0	28·6	25·1	21·4	17·7	13·8	9·9	6·0	2·0	0
23000	59·8	59·7	59·2	58·5	57·5	56·2	54·6	52·8	50·7	48·4	45·8	43·0	40·0	36·8	33·4	29·9	26·2	22·4	18·5	14·5	10·4	6·2	2·1	0
24000	62·4	62·2	61·8	61·0	60·0	58·6	57·0	55·1	52·9	50·5	47·8	44·9	41·8	38·4	34·9	31·2	27·4	23·4	19·3	15·1	10·8	6·5	2·2	0
25000	65·0	64·8	64·4	63·6	62·5	61·1	59·4	57·4	55·1	52·6	49·8	46·8	43·5	40·0	36·3	32·5	28·5	24·3	20·1	15·7	11·3	6·8	2·3	0

IV. CORRECTION FOR			V. CORRECTION FOR THE HEIGHT OF THE LOWER STATION.—*Positive.*							VI. HEIGHT OF A COLUMN OF AIR CORRESPONDING TO ONE TENTH OF AN INCH IN THE BAROMETER.										
Approximate difference of level.	DECREASE OF GRAVITY ON A VERTICAL. *Positive.*		HEIGHT OF THE BAROMETER, IN ENGLISH INCHES, AT LOWER STATION.							Barometer reading in Eng. inches.	TEMPERATURE OF THE AIR, FAHRENHEIT, BEING									
	0	+500	16	18	20	22	24	26	28		40°	45°	50°	55°	60°	65°	70°	75°	80°	85°
Eng. ft.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.		Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
1000	2·5	3·9	1·6	1·3	1·0	0·8	0·6	0·4	0·2	18·5	144·6	146·1	147·7	149·3	150·9	152·5	154·0	155·7	157·2	158·8
2000	5·2	6·6	3·1	2·5	2·0	1·5	1·1	0·7	0·3	19·0	140·8	142·3	143·8	145·4	146·9	148·4	150·0	151·5	153·1	154·6
3000	7·9	9·3	4·7	3·8	3·0	2·3	1·7	1·1	0·5	19·5	137·1	138·6	140·1	141·6	143·1	144·6	146·1	147·6	149·1	150·6
4000	10·8	12·2	6·3	5·1	4·0	3·1	2·2	1·4	0·7	20·0	133·7	135·2	136·6	138·1	139·6	141·0	142·5	143·9	145·4	146·9
5000	13·7	15·2	7·8	6·4	5·0	3·8	2·8	1·8	0·8	20·5	130·5	131·9	133·3	134·7	136·1	137·6	139·0	140·4	141·8	143·3
6000	16·7	18·3	9·4	7·6	6·0	4·6	3·3	2·1	1·0	21·0	127·3	128·7	130·1	131·5	132·9	134·3	135·7	137·0	138·4	139·8
7000	19·9	21·5	11·0	8·9	7·1	5·4	3·9	2·5	1·2	21·5	124·3	125·7	127·0	128·4	129·7	131·1	132·4	133·8	135·1	136·5
8000	23·1	24·7	12·5	10·2	8·1	6·2	4·4	2·8	1·3	22·0	121·5	122·9	124·2	125·5	126·8	128·2	129·5	130·8	132·2	133·5
9000	26·4	28·1	14·1	11·4	9·1	6·9	5·0	3·2	1·5	22·5	118·8	120·1	121·4	122·7	124·0	125·3	126·6	127·9	129·2	130·5
10000	29·8	31·5	15·7	12·7	10·1	7·7	5·5	3·5	1·7	23·0	116·2	117·5	118·8	120·0	121·3	122·6	123·8	125·1	126·4	127·7
11000	33·3	35·1	17·2	14·0	11·1	8·5	6·1	3·9	1·8	23·5	113·7	115·0	116·2	117·5	118·7	120·0	121·2	122·5	123·7	124·9
12000	36·9	38·7	18·8	15·3	12·1	9·2	6·6	4·2	2·0	24·0	111·3	112·6	113·8	115·0	116·2	117·4	118·6	119·9	121·1	122·3
13000	40·6	42·5	20·4	16·5	13·1	10·0	7·2	4·6	2·2	24·5	109·1	110·3	111·5	112·6	113·8	115·0	116·2	117·3	118·6	119·8
14000	44·4	46·3	21·9	17·8	14·1	10·8	7·7	4·9	2·3	25·0	106·9	108·1	109·3	110·4	111·6	112·8	113·9	115·1	116·3	117·4
15000	48·3	50·3	23·5	19·1	15·1	11·5	8·3	5·3	2·5	25·5	104·8	105·9	107·1	108·2	109·3	110·5	111·6	112·8	113·9	115·1
16000	52·3	54·3	25·1	20·3	16·1	12·3	8·8	5·6	2·7	26·0	102·7	103·9	105·0	106·1	107·2	108·4	109·5	110·6	111·7	112·8
17000	56·4	58·4	26·6	21·6	17·1	13·1	9·4	6·0	2·8	26·5	100·9	102·0	103·1	104·2	105·3	106·4	107·5	108·6	109·7	110·8
18000	60·5	62·6	28·2	22·9	18·1	13·8	9·9	6·3	3·0	27·0	99·0	100·1	101·2	102·3	103·3	104·4	105·5	106·6	107·6	108·7
19000	64·8	67·0	29·8	24·1	19·2	14·6	10·5	6·7	3·2	27·5	97·2	98·2	99·3	100·3	101·4	102·5	103·5	104·6	105·6	106·7
20000	69·2	71·4	31·3	25·4	20·2	15·4	11·0	7·0	3·3	28·0	95·4	96·5	97·5	98·6	99·6	100·7	101·7	102·8	103·8	104·8
21000	73·6	75·9	32·9	26·7	21·2	16·1	11·6	7·4	3·5	28·5	93·8	94·8	95·8	96·9	97·9	98·9	99·9	100·9	101·9	103·0
22000	78·2	80·5	34·5	28·0	22·2	16·9	12·1	7·7	3·7	29·0	92·1	93·1	94·1	95·1	96·2	97·2	98·2	99·2	100·2	101·2
23000	82·9	85·2	36·0	29·2	23·2	17·7	12·7	8·1	3·8	29·5	90·6	91·6	92·6	93·6	94·5	95·5	96·5	97·5	98·5	99·5
24000	87·6	90·0	37·6	30·5	24·2	18·5	13·2	8·4	4·0	30·0	89·1	90·0	91·0	92·0	92·9	93·9	94·9	95·9	96·8	97·8
25000	92·5	94·9	39·1	31·8	25·2	19·2	13·8	8·8	4·1	30·5	87·6	88·5	89·5	90·4	91·4	92·3	93·3	94·2	95·2	96·1

769. French Formulas. French barometers are graduated in French millimetres, each = 0·03937 inch, and the thermometer is centigrade, in which the freezing point is zero and boiling point 100°:

$$a^\circ \text{ C.} = (\tfrac{9}{5}a + 32)^\circ \text{ F.}$$

Then, the French formula corresponding to [1] is the following (H and h' being in millimetres, and the temperatures centigrade):

$$h = h'\left(1 + \frac{T - T'}{6200}\right)$$

And the difference of heights in metres

$$= 18336\,(\log.\,H - \log.\,h)\left\{\begin{matrix}\left(1 + \dfrac{2\,(t + t')}{1000}\right) \\ (1 + 0{\cdot}00265\,.\,\cos.\,2\,L) \\ \left(\dfrac{1 + x' + 15926}{6372481}\right) + \dfrac{S}{3186241}\end{matrix}\right\} \qquad [2.]$$

770. Babinet's Simplified Formula, without Logarithms.

h' is reduced to h, as before, viz.: $h = h'\left(1 + \dfrac{T - T_1}{6200}\right)$

Then, the difference of heights in metres

$$= 16000\,.\,\frac{H - h}{H + h}\left(1 + \frac{2\,(t + t')}{1000}\right) \qquad [3.]$$

The heights are in millimetres, and the temperatures centigrade.

Example. $H = 755. \quad h = 745$

$t = 15^\circ \quad t' = 10^\circ$

$$\text{Ht.} = 16000\,\frac{10}{1500}\left(1 + \frac{50}{1000}\right) = 112 \text{ m.}$$

Correct result is 111·6 m.

This formula is a very near approximation for moderate heights.

Babinet's formula in English measures (the heights being in inches, and temperatures Fahrenheit) is in feet:

$$52494\left(\frac{H - h}{H + h}\right)\quad\left(1 + \frac{t + t' - 64}{900}\right) \qquad [4.]$$

Leslie's formula is:

$$\text{Height in feet} = 55000\,\frac{B - b}{B + b} \qquad [5.]$$

In which B = upper reading, and b = lower reading. This is for a temperature of 55° F.

771. Approximations. One tenth of an inch difference of readings in two places corresponds to about ninety feet difference of elevation. One millimetre difference of readings corresponds to about ten and a half metres difference of height, or about thirty-four feet.

This is correct near the freezing point, and near the level of the sea. The height corresponding to any given difference of readings increases, however, with the temperature and with the height of the station. Thus, at 70° F. $\frac{1}{10}$ of an inch corresponds to an elevation of 95 feet; and one millimetre at 30° C. corresponds to $11\frac{3}{4}$ metres, or about 40 feet.

772. Accuracy of Barometric Leveling. Barometric observations, carried on over considerable periods of time have shown that there are fluctuations of atmospheric pressure, some of which are periodic and some irregular. There is a variation of pressure each day, the pressure being greatest about ten o'clock and four o'clock, both morning and afternoon. There is a variation also with the seasons. The amount of moisture in the air affects the height of the barometric column, and there are many irregular variations from unknown causes.

The results of a single observation can not be relied upon to give accurate results. When it is possible, the readings of the barometer should be taken hourly during the whole day and a mean of the results taken. Still better results can be obtained by continuing the observations over a longer period. The accuracy of the results increases with the number of repetitions of observations, the mean of them being taken. An experienced observer, by sufficient repetitions, can determine heights to within a few feet.

PROFESSOR GUYOT'S RESULTS.

HEIGHTS FOUND BY THE BAROMETER.	CORRESPONDING HEIGHTS FOUND BY THE SPIRIT LEVEL.
6707 feet.	6711 feet.
2752 "	2752 "
6291 "	6285 "
	6293 "

The observations at the two places, whose difference of heights is to be determined, should be taken simultaneously at a series of intervals previously agreed upon, the distance apart of the places being as short as possible. Distant places should be connected by a series of intermediate ones.

References.—"The Use of the Barometer," by R. S. Williamson. Tables, Discussions, and Directions in "Smithsonian Miscellaneous Collections," vol. i; "United States Coast and Geodetic Survey Report," 1881, Appendix 10.

CHAPTER XVI.

TOPOGRAPHY.

773. Definition. Topography is the complete determination and representation of any portion of the surface of the earth, embracing the relative position and heights of its inequalities: its hills and hollows, its ridges and valleys, level plains, slopes, etc., telling precisely where any point is, and how high it is.

It therefore determines the three co-ordinates of any point; the horizontal ones by surveying, and the vertical ones by leveling.

The results of these determinations are represented in a conventional manner, which is called "topographic mapping."

The difficulty is, that we see hills and hollows in *elevation*, while we have to represent them in *plan*.

The instruments and methods employed in topographic surveying, and the system or systems employed in topographic mapping, depend upon the purpose for which the map is made and the accuracy required. But to whatever use the map is to be put, and by whatever method the configuration of the surface is to be represented, it may be said, in general, that the object of a topographic survey is to obtain the necessary data from which a contour map of the region may be made.

774. Division of Work. The work of topography is divided into two parts:

1. Field work, reconnaissance and topographic surveying.
2. Office work.

The *office work* is divided into two parts:

1. The reduction of the notes and the adjustment of the errors.
2. Topographic mapping, which is the representation of the

natural and artificial features of the region to be mapped by conventional signs.

775. Field Work. *A Reconnaissance*, which is a careful preliminary examination of the region to be surveyed, should always be made before commencing the actual survey. In railroad work the reconnaissance is made by the locating engineer, and is for the purpose of selecting a line along which the topography is afterward taken. The basis of operation or "control" of a topographic survey may be a triangulation, or a series of connecting lines, carefully run through the region for that purpose, called a "traverse."

If the control of the topographic map is to be a triangulation, the reconnaissance is made by the triangulation party for the purpose of locating the triangulation stations. The location of these stations should be made with reference to the needs of the topographers, as far as may be, and still maintain the accuracy of the control. If the topographic map is to be controlled by traverses, the reconnaissance should be made by the topographer.

TOPOGRAPHIC SURVEYING is the complete determination of all features, natural and artificial, of a region. This comprises their *geographical position*—that is, their location on the surface of the earth, considered as a plane or a sphere, with reference to some origin and meridian, so that they may be relocated in the same relative position on the map; their *elevation*, or the distances above or below some common plane of reference; the character of the surface, and the vegetation in its natural or cultivated state.

776. Methods for making the Survey. There are several methods employed:

1. Surveying and leveling the triangulation skeleton and its traverses.

2. Surveying and leveling profile lines intersecting the area.

3. Surveying each contour line or contours at sufficient intervals to enable others to be interpolated.

4. The division of the area into squares, parallelograms, or triangles.

5. Determining the position and elevation of a sufficient number

of characteristic points of the area, so that contours may be interpolated. This is the one usually employed.

According to the instruments used, there are several methods of taking topography:

1. With compass or transit, chain or tape, and level.
2. With the stadia.
3. With the plane table.
4. With the clinometer or hand level.
5. By photography.

The first three of these are more properly means for topographic surveying, while the last two are means for topographic sketching, the results being less accurate. If the surface is to be determined accurately enough for the computation of earthwork, the first method should be used. If general topography is to be taken over large areas, the control of the survey being a triangulation, or traverse lines, then the second or third method should be used. If the topography is to be sketched from a line, as in railroad location, the fourth method may be used.

If the topography is to be sketched for military purposes, or for a map to show the general relief of the ground, then the fifth method may be used.

777. Cross Sectioning. The area is divided into squares, using transit or compass and tape or chain. Levels are taken at intersections, which are marked by stakes.

This method must be adopted if the results are to be used in the computation of earthwork, and should be employed from the beginning, if the exact location of the earthwork is known in advance.

If the location of the work of construction is to be determined by the survey, then, on account of the greater cost of this method, it is more economical to make a general survey by one of the other methods for the purpose of locating the work, afterward cross sectioning only the part necessary.

Base of Operation.—The section lines should be located with reference to some base. If a field, use one of its sides; if for a road, use its center line. If an extended area is to be cross sectioned, the north-and-south line is preferably used, as it facilitates the exten-

sion of the lines through thickly wooded places by means of the compass.

Instruments.—The instruments required are: Transit or compass for running out the lines, a chain or tape and a sight rod, a level and rod, or a transit with stadia wires, and a stadia rod to take the elevations of the points. Two or three men are necessary.

Method of Work.—Having selected a base, it is divided into equal parts, their length being that desired for the sides of the squares. The squares should be 50 to 100 feet on a side for preliminary work, and 25 to 50 feet for construction, depending upon the irregularity of the ground. Squares 27 feet on a side have been used with the object of facilitating the reduction of the earthwork to cubic yards. Any such advantage is, however, much more than counterbalanced by the inconvenience in the field work, and in making and using the map.

By means of the transit or compass, lines are run out at right angles to the base line, stakes being set at intervals equal to those on the base line. If the party is composed of three men, one of them lines in each stake from the instrument, while the other two measure off the distances and mark and set stakes at the points determined, the head chainman marking and setting the stakes.

The work can, however, be done by two men. A sight rod being first set upon the line at a considerable distance from the instrument, the rear chainman can then line in the stakes with sufficient accuracy by means of the sight rod.

If the party consists of four men, it is the duty of one man to mark and drive stakes.

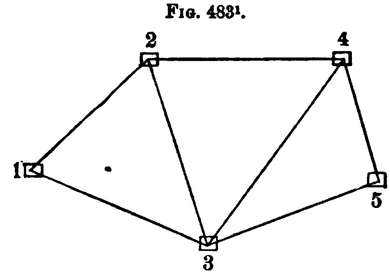
For the purpose of designating the points, the squares may be considered to be formed by two systems of parallel lines intersecting each other at right angles. The lines of one system are lettered consecutively, those of the other are numbered. Any point is then designated by the letter and number of the two lines intersecting them. Thus, *c* 3 is the point at the intersection of the lines marked *c* and 3. Objects are located by tying them to two of the stakes. Three men can set stakes and level six acres per day over broken ground, setting stakes every 50 feet and tying in trees.

778. The Stadia Method (Art. 346–350, Part I). When topography is to be taken over large areas, as in the solution of questions of drainage or water supply, in geological surveys, and preliminary surveys for park work, the stadia method is preferred. Engineers who have used it for preliminary railroad surveys advocate its use for that purpose. In this method, points distributed over the area to be surveyed are determined by polar co-ordinates. The distance and elevation of the point from the instrument is measured by the stadia, and the angle is measured on the horizontal circle of the transit.

Base of Operation or Control.—The control for large areas, or for small areas where great accuracy is required, should be a triangulation (Chaps. X and XI). Where a heavily wooded country makes a triangulation impractical, on account of cost, the control may be obtained through primary traverses.

Where exact positions of points and boundaries are not required the stadia may be used alone, in which case control may be had by closing upon the starting point, or by carrying the work forward in the form of a chain of triangles, which may be accomplished by locating, from the starting point, two station points that are intervisible, and after that selecting each succeeding station point so that the distances to the station occupied and the preceding one may be read. In Fig. 483' stations 2 and 3 are located from 1, so that the distance from 2 to 3 can be read with the stadia; 4 is located from 2, the distance from 3 to 4 being capable of measurement by stadia.

Fig. 483'.



779. The Transit. There are certain points to be observed in the construction of a transit that it may be best fitted for this work.

1. The horizontal limb should read to 30 seconds, and should have the graduations numbered from 0° to 360° in the direction of the movement of the hands of a watch.

2. The instrument should have a full vertical circle rigidly attached to the telescope axis, graduated from 0° to 90° in both directions to read to the same angle as the horizontal circle—i. e., 30

seconds. The vertical circle should be provided with two double verniers attached to a horizontal vernier arm, the zeros of the two verniers being in a horizontal plane when the instrument is leveled up. The horizontal vernier arm should have a level attached, of a sensitiveness corresponding to the angle read by the verniers. It is upon this level that dependence will be placed for the accuracy of the levels, and not upon the levels upon the horizontal vernier plate.

3. The stadia wires should be fixed, to avoid the introduction of errors due to the wire interval varying, and to permit the placing of both systems of wires in the same plane, so that the eyepiece may focus on both alike.

4. The telescope should be inverting, as it permits a longer focal distance, and in consequence gives a flatter field for the same length of tube, and the removal of the lenses for erecting the image gives a better illuminated and more sharply defined image.

5. The bubbles on the plate should be delicate and stable if the carrying of the line of levels depends upon them.

6. For convenience in centering the instruments over station points, it should have a shifting head.

7. A solar attachment and magnetic needle will be valuable attachments for the purpose of checking the azimuth as the line proceeds.

780. The Stadia Rod. Stadia rods should be clear, well-seasoned white pine, seven eighths of an inch thick, four to five inches wide, ten to sixteen feet long, and protected by a metal shoe to keep the lower end from becoming split or battered. The rod may be stiffened by having a piece along the back. It may be hinged at the center to facilitate its transportation and to protect the graduations, a bolt on the back holding it in position when in use. Before being graduated it should receive a sufficient number of coats of white paint to make it thoroughly white, a dead white without gloss. This result is obtained by using but little oil. Targets are sometimes used on them. To insure that the rod is held in a vertical position a circular or other form of rod level is attached. A self-reading level rod may be used for short distances, if the wires are adjustable or the wire interval has been determined in standard units.

Adjusting the Wires.—The usual method of adjusting the wires for a given rod, or the determination of the wire interval, has been given in Art. 349, Part I. Some engineers advise that these determinations be made at noon, as being the time of least refraction of the sun's rays; others, that a cloudy day with a clear atmosphere be employed. But when such has been the case it has been found that a systematic error was introduced in the subsequent work.

The wire interval obtained will depend upon the atmospheric condition and the nature of the surface over which the sight is taken. It is evident, therefore, that for the best results the above conditions at the time the wire interval is obtained or stadia board graduated should be the same as those that will be encountered upon the work. The condition of the atmosphere usually changes with the hour of the day, therefore if field work is to be carried on throughout the day the wire interval to be used should be one obtained by taking the average of a series of determinations made at intervals through a day, selecting a day of average atmospheric conditions, and in the same season of the year that the work is to be done.

For city work the wire interval should be determined along a street, while for suburban work the determination should be made upon ground similar to that to be encountered.

It has been shown that the air within a metre of the ground may be so heated by reflection of the sun's rays as to materially affect results. It is to the *differences* in the powers of refraction of these heated strata that is due the systematic errors introduced in stadia work by the use of a wire interval determined under a single condition of atmosphere and surface.*

The error due to this difference in refraction may be lessened by reading the distance in the upper half of the board, using the interval between each stadia wire and the center wire in turn and adding them. If the work is of such a nature as to introduce entirely new conditions the wire interval should be redetermined.

Having the determination of the wire interval under a number

* "An Experimental Study of Field Methods which will insure to Stadia Measurements greatly Increased Accuracy," by L. S. Smith, "Bulletin of the University of Wisconsin," Eng. Series, vol. i, No. 5.

of conditions, it would be but little additional work to note the conditions or number of wire interval to use in the reduction of the notes, which could be made either with the slide rule or tables.*

For the determination of the wire interval use a rod with two targets of cardboard, one fixed and the other movable. Let the horizontal line of the movable target be a slot cut out so that when its position is determined the rod can be marked.

To eliminate the personal equation of the observer the wire interval should be determined by the man who is to make the observations in the field.

A level should be used with the rod, to insure its being held vertical. Part I, page 376.

Graduation of the Stadia Rod.—The graduation of the stadia rod should be simple and clear, having distinctive unit divisions and subdivisions, the unit divisions being preferably numbered. Where not numbered the graduations should be symmetrical with reference to the two ends. At no point should the rod be black across the entire face. There should be some white background for the wire at all points.

For general work, the design of the graduations should be such as to give short distances accurately and still enable the observer to read long distances readily. For a given piece of work, however, this dual capacity is not so important, since the readings for which the great accuracy is desired can probably be taken over a range small enough to permit one to graduate the rod either for long distances or short distances only.

The unit used in graduating a stadia rod is influenced by the average length of distances that are to be read, the metre being the preferable unit for long distances and the foot for short ones. The design of the graduations will in turn depend upon whether the unit employed is the foot or the metre, a type of graduation for one not being suitable for the other.

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* A table for this purpose, due to Mr. P. D. Cunningham, is given in the "Transactions of the American Society of Civil Engineers," vol. xxxiv, No. 4, October, 1895. Van Ornum, "Topography on the Survey of the Mexico-United States Boundary."

Stadia Rods.—Figs. 484 and 485 show two types of graduation suitable where the metre is the unit. Fig. 484 was used by the

FIG. 484.

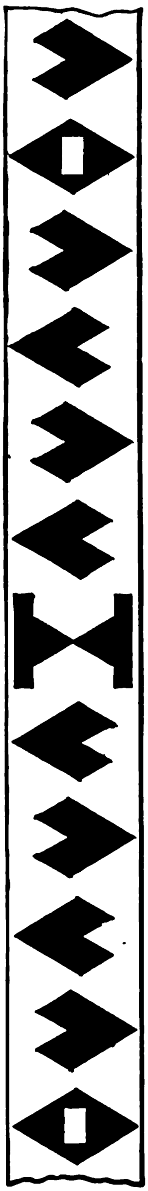
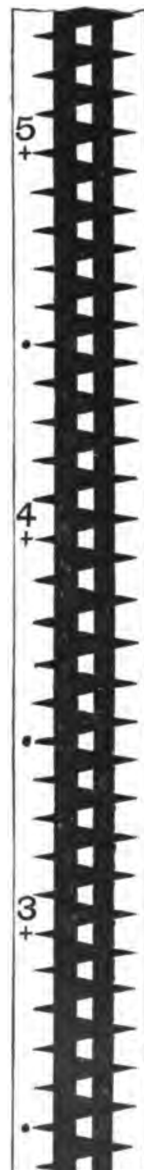


FIG. 485.

FIG. 486.

FIG. 487.



United States Coast and Geodetic Survey. The angles of the graduations divide the rod to two-centimetre intervals.

Fig. 485 was used on the recent survey of the Mexican border. Its graduation needs no explanation. Surfaces shaded were painted red.

Figs. 486 and 487 are types suitable where the foot is the unit. In Fig. 487 the width comprised between the ends of the points is divided into five equal parts, the vertical black lines taking up two of these divisions. The diagonals then give 100ths of a foot, and permit reading direct to single feet.

Instead of graduating the rod so that at a distance of $100 + (f + c)$, the unit, will be exactly intercepted upon the rod (Part I, page 240), a plan of graduation made to facilitate the reduction of the notes, the division of the rod into true units is preferable for the following reasons: 1. That the distance of the center wire above the ground at the time of reading the vertical angle may be known in true units. 2. If different values of the wire interval are to be used for different conditions it would be useless to graduate for one value. 3. It is convenient to determine elevations by direct leveling.

If a rod is graduated to units determined in terms of the wire interval the reverse side of it may be graduated to true units for use in direct leveling. The two styles of graduation must then be entirely different in order to avoid confusion.

781. Organization of Party. The organization of a party and general methods of work will depend upon the nature of the country traversed and of the results desired. Changes in the make-up of parties or methods of work, as given below, will suggest themselves for any special work.

The method of work will depend upon whether the contour interval in the map is to be large or small, since for large intervals the topography can be sketched in from controlling points, while if it is small, frequent points for the interpolation of contours must be taken.

1. For small contour interval or limited area.

For economy and speed, the party for taking topography with

stadia will consist of a transitman or observer, a recorder in charge of the notebook, who should also be capable of making such sketches as are necessary, and two to four men with stadia rods. The greater the distances to be traversed by the stadia men between points taken, the greater number the observer can work to advantage. One or two axemen may be employed if clearing can be done with economy.

The party may be reduced to two men: one to handle the instrument, record notes, and make sketches, the second to carry the rod.

782. Method of Work. The instrument is first set up over a triangulation station (marked △) or a traverse station (marked ⊡), and clamped in such a position that when it points to any of the triangulation or traverse stations visible from the station occupied, the reading of the vernier will give the observed or computed azimuth of the line from the station occupied to the other station.

The upper movement is then unclamped, and all azimuths read will be referred to the meridian used in the triangulation.

Points are then selected to be located by the stadia. These points should be chosen so as to give the greatest amount of information. Such points are changes of slope, tops of ridges and knolls, bottoms of ravines, intersection of street or property lines, etc.

The stadia rod is held vertically at the point to be located and of which the elevation is to be obtained. The following order in making the observations will facilitate the work:

1. Read distance.
2. Set center wire upon the height of instrument indicated upon the board.
3. Signal stadia man to go to next point to be taken.
4. Read vertical angle and azimuth.

Although the formula for reduction of elevations and distances is based upon the supposition that when reading the distance the center wire is placed at a point upon the rod equal to the height of instrument, a change of one half the unit in either direction to

place the lower wire upon an even division, leaving but one wire to be read, will introduce an error so small that it may be neglected in work of the highest grade, excepting lines connecting instrument points.

For measuring the vertical angle, the center wire must be set upon the rod either at the height of instrument or at some height which must be recorded, and from which the elevation of the station can be computed. Since the positions and elevations of stadia stations are usually determined by stadia alone, the vertical angles and distances between stadia stations should be read as a check forward and backward from each station.

It is advisable to locate the next stadia station at once upon occupying a new station, not waiting until all the topography to be taken from it has been taken. The reasons are: 1. If an instrument stands for some time in the sun it twists around in the direction of the motion of the hands of a watch. This would introduce the largest error in the line whose azimuth is of most importance. 2. In general, the topography over one half the interval between two stations should be taken from each. If the location of the next station is not selected at once, one is liable to take topography in the vicinity of what will be the location of the next station.

Upon occupying the second station the readings of the verniers are to be interchanged, or, if the instrument has but one vernier, 180° is to be added to its reading when the station now occupied was located, and the vernier clamped before taking the backsight. Clamp the lower motion with the instrument pointing to the preceding station; then unclamp the upper movement and proceed as before. By following this method all azimuths read will be referred to the original meridian, and any error will be avoided that would be introduced by transiting the instrument if the line of collimation was out of adjustment.

If some tall object, either within the survey or at one side of it, is visible from a number of stations, it is advisable to take azimuths to it from all such stations, as they may be the means of locating an error in azimuth made at some station. If the needle is read at each station it will also be a check upon errors in the field.

To close a stadia meander: Set the transit upon the triangula-

tion or traverse station to which the distance and vertical angle was read from the last meander station. Having set the verniers properly, read the distance and vertical angle back to the last stadia station. Turn the instrument until it points to some other triangulation or traverse station. Then the reading of the vernier should be the same as the computed azimuth of this line.

To tell whether the meander closes in distance, the line must be either plotted or the co-ordinates of the station upon which the meander was closed must be computed by means of the meander. They should agree with the co-ordinates as determined by the triangulation.

Where an accurate determination of azimuth is necessary, as in sighting to any instrument point, to orient the instrument, or in locating a new meander station, the edge of the stadia rod should be turned toward the instrument.

Either the observer or recorder should make an accurate sketch of the area surveyed around each station. If the man who makes the survey is to make the map also, and he is sufficiently experienced to recall the topographic features, he may omit the sketch.

The area of which the topography is desired is so covered with a series of meanders as to permit of all points being reached with the stadia.

783. Form of Notes. The notebook should have six columns upon the left-hand page, the right-hand page being quadrille ruled to facilitate accurate sketching.

At the beginning of a set of notes upon a new piece of work, record at the top of the left-hand page the nature of the work to which the notes to follow refer. Record the height of instrument, and record prominently the station at which the instrument is located. If the notes from a single station do not completely fill a page, it is usually better, on account of the clearness of notes and sketches, to begin the notes for a new station upon a new page. In the first column, which should be wider than the others, is recorded a description of the different points taken. These points should be lettered or numbered consecutively. The numbers should be marked upon the sketch in their relative positions.

In the second column record azimuths; in the third, distance; in the fourth, vertical angles. The fifth column is reserved for the computed differences in elevation, and the sixth column for elevations. The corrected distances are placed in the same column with the distances read. Above the sixth column should be left space for recording the elevation of the station at which the notes were taken.

At the top of the right-hand page record the date, members of party and their positions, name and number of instrument, and number of rod.

These last two points are of importance only when instruments with different intervals or rods with different divisions might have been used.

Record state of weather, as affecting results.

The first point sighted to will always be some previously determined station, therefore on the first line should always be found a station number, with its azimuth from the station occupied. This azimuth is set off on the instrument before taking the backsight, the distance and vertical angle being then read as a check if the station occupied is a meander station.

The Sketch.—The sketch on the right-hand page should show all topographic features, with numbers at points where the rod was held. Locate the instrument station upon the page for the sketch, with reference to the distance to which topography is to be taken from it in different directions. The previous station and the station next to be occupied should always be noted in position on the sketch if the scale permit, or their directions indicated by arrows if to scale they fall outside the sketch. The direction of contours should be shown, in order that any irregularities in the surface may be properly mapped. Surface conditions should be noted, as woods, cultivated meadow, etc., or small patches may be marked with conventional signs.

If a small contour interval is not desired, or else not attainable, the points determined by stadia, other than those used as instrument points for carrying forward the meander, are used as bases for sketching the topography in their vicinity.

The topographer takes a sketching board and a sheet upon

which all such points with their elevations are plotted. He sketches in all the actual contours from the country as copy, making the map complete, except inking and lettering. However, where contours fall so close together that the intermediate ones may be interpolated afterward, to save time every fourth or fifth one only need be drawn in the field.*

784. Reduction of Stadia Notes. The formulas for the reduction of stadia readings to obtain the horizontal distance and height, or difference in elevation, are derived in Part I, Arts. 346–350.

They are:
$$\text{A B} = \frac{f}{a} k \cos.^2 e + (f + c) \cos. e \qquad [5.]$$

$$\text{B D} = \frac{f}{a} k \tfrac{1}{2} \sin. 2 e + (f + c) \sin. e \qquad [6.]$$

In which A B is the horizontal distance from the center of instrument to the point at which the rod was held, B D is the difference in elevation between the two points, a is the distance between the stadia wires, k is the space intercepted upon the stadia rod, f is the principal focal distance, measured upon the telescope from the objective when the telescope is focused upon a distant object, c is the average distance from the objective to the center of the instrument, a quantity varying within narrow limits but taken as constant.

To determine $\left(\frac{f}{a}\right)$, which is called the interval factor, for any instrument, measure off from the instrument a level base of length, B, hold the stadia rod at the farther end and let the reading be k'.

* In the work upon the Mexican boundary, where topography was taken for 4,000 metres on either side of the line, signals upon commanding peaks were located by intersections from the stadia stations along the boundary, and their elevations determined by measuring the vertical angles to them. Transits were in turn placed upon two of these peaks at a time. An engineer and rodman, both mounted when the country would permit, selected points upon which the rod was held at the same time the observers were signaled. The point was located by intersections, and its elevation determined by measuring the vertical angle. The time of taking each observation was recorded by all three parties, to have a check upon corresponding intersections. The engineer sketched the topography around these points, reducing it afterward to true elevation as determined by the elevation of the point. Sights were taken to 10,000 and 12,000 metres, with 80 per cent of the elevations agreeing within three metres.

Then [5] becomes $\quad B = \frac{f}{a} k' + (f + c)$

$$\text{or } \left(\frac{f}{a}\right) = \left(\frac{B - f - c}{k}\right)$$

substituting this value, [5] and [6] become

$$A\,B = \left(\frac{B - f - c}{k'}\right) k \cos.^2 e + (f + c) \cos. e \qquad [7.]$$

$$B\,D = \left(\frac{B - f - c}{k'}\right) k \tfrac{1}{2} \sin. 2\,e + (f + c) \sin. e \qquad [8.]$$

which are Prof. S. W. Robinson's formulas.

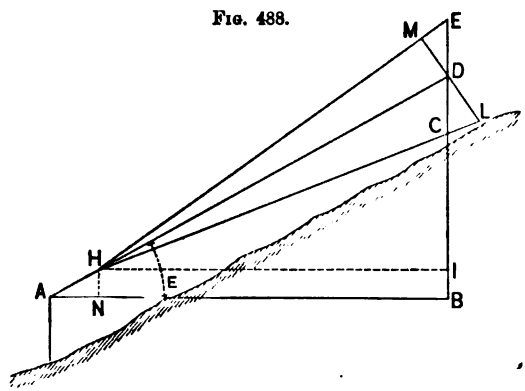
If the base measured from the instrument was equal to some even number of hundred units B', + $(f + c)$, the formulas would reduce to

$$A\,B = \frac{B'}{k'} . \, k \cos.^2 e + (f + c) \cos. e \qquad [9.]$$

$$B\,D = \frac{B'}{k'} k \tfrac{1}{2} \sin. 2\,e + (f + c) \sin. e \qquad [10.]$$

If k' is expressed in true units, then $\frac{B'}{k'}$ gives the interval factor to be used in the reduction of all readings, the rod being graduated to true units. If the interval k' be divided into the same number of units as there were hundreds of units in B', each unit intercepted will for any reading represent one hundred units, and the formulas may be written

FIG. 488.



$$A\,B = R \cos.^2 e + (f + c) \cos. e \qquad [11.]$$

$$B\,D = R \tfrac{1}{2} \sin. 2\,e + (f + c) \sin. e \qquad [12.]$$

where R is the rod reading multiplied by 100.

The formula for the reduction of inclined sights, given in Part I, page 237, while much used, is not rigidly correct. The error arises

from taking M L = E C cos. e—that is, assuming E M D and D L C both right angles. The correct expression is developed as follows:

Let $m = \frac{1}{2}$ the visual angle, assuming wires equidistant.

$$\begin{aligned} \text{Then the angle H C D} &= 90 + e - m \\ \text{H M D} &= 90 - m = \text{H L D} \\ \text{M E D} &= 90 - e - m \\ \text{D C L} &= 90 - e + m \\ \text{M H L} &= 2\,m \\ \text{E D M} &= \text{C D L} = e \end{aligned}$$

$$\text{H D} = \frac{\text{D E sin. E}}{\text{sin. } m} = \frac{\text{D E cos. } (e+m)}{\text{sin. } m} \qquad [13.]$$

$$\text{H D} = \frac{\text{C D sin. } c}{\text{sin. } m} = \frac{\text{C D cos. } (e-m)}{\text{sin. } m} \qquad [14.]$$

$$\text{H D} = \frac{\text{M L}}{2 \text{ tan. } m} = \frac{\text{M L cos. } m}{2 \text{ sin. } m} \qquad [15.]$$

From [13] and [15] $\qquad \text{D E} = \dfrac{\text{M L cos. } m}{2 \text{ cos. } (e+m)}$ [16.]

" [14] " [15] $\qquad \text{C D} = \dfrac{\text{M L cos. } m}{2 \text{ cos. } (e-m)}$ [17.]

" [16] " [17]

$$\text{C D} + \text{D E} = \text{C E} = \frac{\text{M L cos. } m}{2}\left[\frac{1}{\text{cos. } (e+m)} + \frac{1}{\text{cos. } (e-m)}\right] \qquad [18.]$$

$$\text{M L} = \text{C E}\left[\frac{\text{cos.}^2 e \text{ cos.}^2 m - \text{sin.}^2 e \text{ sin.}^2 m}{\text{cos. } e \text{ cos.}^2 m}\right]$$

$$= \text{C E}\left[\text{cos. } e - \frac{\text{sin.}^2 e}{\text{cos. } e} \text{tan.}^2 m\right] \qquad [19.]$$

$$= \text{C E cos. } e - \text{C E}\frac{\text{sin.}^2 e}{\text{cos. } e}\text{tan.}^2 m$$

But $\text{C E}\dfrac{\text{sin.}^2 e}{\text{cos. } e}\text{tan.}^2 m = 0$ approx. [20.]

e. g., $m = 30'$ $\quad e = 10°$ $\quad$ C E = 1,000 the value of [20] = ·0023
$m = 30'$ $\quad e = 30°$ $\quad$ C E = 500 " " " = ·012

and the formula may be written,

$$\text{M L} = \text{C E cos. } e. \qquad [21.]$$

Stadia notes are reduced with the aid of either (1) tables, (2) graphic diagrams, or (3) slide rules constructed for that purpose.

The values of f and c being peculiar to each instrument, and

depending upon the unit, must be determined for each instrument, and be expressed in the units of the work.

It is only necessary to determine the values of $(f+c)$ cos. e and $(f+c)$ sin. e for a comparatively few values of e depending upon the degree of precision sought. If elevations are to be computed to the nearest tenth, determine the values of e, $e_{,}$, $e_{,,}$, $e_{,,,}$, etc., for which

$$(f+c)\ \text{sin.}\ e_{,} \ = \cdot 05$$
$$(f+c)\ \text{sin.}\ e_{,,} \ = \cdot 15$$
$$(f+c)\ \text{sin.}\ e_{,,,} = \cdot 25,\ \text{etc.}$$

Then, for all values of e between o and $e_{,}$, $(f+c)$ sin. $e = o$; for values of e between $e_{,}$ and $e_{,,}$, $(f+c)$ sin. e will be taken $= 0{\cdot}1$, etc.

Still fewer values of e for $(f+c)$ cos. e will be needed, since for most of the angles read in general practice the cosine is so near unity that $(f+c)$ will be added to get distances.

These limiting values of e should be tabulated to use with all the methods for reducing the terms R cos.^{2} e and R $\frac{1}{2}$ sin. 2 e, which, depending only upon the rod reading and the vertical angle, may be arranged in tables or used as the basis of graphic diagrams or slide rules, which can be used with any instrument and with any unit of length.

Equations 7 and 8 may be written as follows, combining the two terms:

$$\text{A B} = (\text{R} + f + c)\ \text{cos.}^2 e\ \text{(approx.)}$$
$$\text{B D} = (\text{R} + f + c)\ \tfrac{1}{2}\ \text{sin.}\ 2\,e \quad \text{"}$$

To show the amount of error introduced by the approximation, take $e = 15°$ and $30°$:

$e = 15°$, cos. $e = \cdot 965$	$e = 30°$, cos. $e = \cdot 866$
cos.^{2} $e = \cdot 933$	cos.^{2} $e = \cdot 750$
cos. $-$ cos.$^2 = \cdot 032$	cos. $-$ cos.$^2 = \cdot 116$
sin. $e = \cdot 258$	sin. $e = \cdot 50$
$\frac{1}{2}$ sin. 2 $e = \cdot 25$	$\frac{1}{2}$ sin. 2 $e = \cdot 433$
sin. $e - \frac{1}{2}$ sin. 2 $e = \cdot 008$	sin. $e - \frac{1}{2}$ sin. 2 $e = \cdot 067$

Then the errors due to the use of the equations in the above form will be:

In distance,	$e = 15°$	$\cdot 032\,(f+c)$	$e = 30°$	$\cdot 116\,(f+c)$
In elevation,	"	$\cdot 008\,(f+c)$	"	$\cdot 069\,(f+c)$

quantities that may be neglected.

785. Tables. The tables given in Part I are explained on pages 238 to 241. The best tables for this work are Jordan's (see page 241, Part I). Some tables give only values of $\sin.^2 e$ and $\frac{1}{2} \sin. 2 e$ to a radius of unity, thereby making it necessary to multiply by R. With the aid of a slide rule these results may be rapidly obtained, (see I. O. Baker's "Surveying Instruments").

Ockerson and Teeple, Assistant United States Engineers, published in 1875 tables based upon Prof. Robinson's formulas, which gave the distance in metres, varying by 10-metre intervals from 100 to 600 metres, and gave differences in elevation in feet. They used a value for $(f + c) = 0{\cdot}43$.

The values of $R \frac{1}{2} \sin. 2 e$ are given for values of e varying by from one to three minute intervals, from 0° to 10° for the longer distances, and from 0° to 30° for the shorter.* By combining two values, or multiplying those between 10 and 100 by 10, any desired distance may be reduced.

A table of values for $(f + c)$ is also given.

786. Graphic Diagrams. The first graphic diagrams for the reduction of notes were made by two Swiss engineers. They were based upon a quadrant, and three different diagrams were necessary for vertical angles up to 18° 26′. Several forms of diagrams have since been designed.

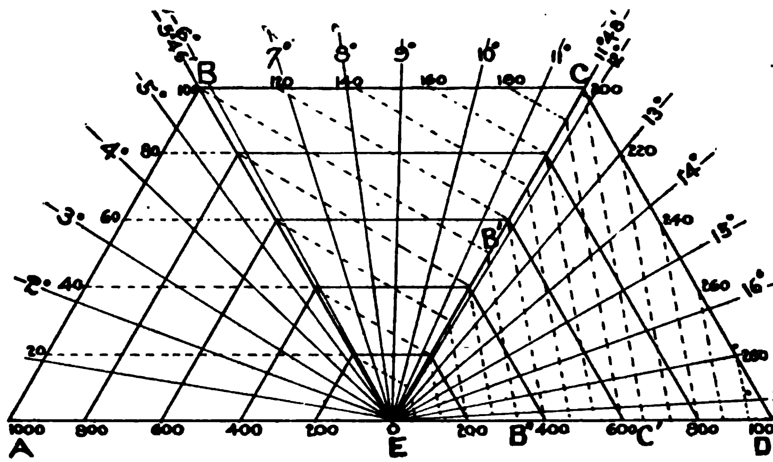
Fig. 489 is one form of these diagrams. It is a trapezoid composed of three equilateral triangles. The changes in direction of the lines bounding the diagram, and on which are read the differences in elevation, are made at radial lines corresponding to vertical angles, for which $\frac{1}{2} \sin. 2 e = {\cdot}1, {\cdot}2$, and ${\cdot}3$.

Construction of a Diagram for the Determination of Differences of Elevation. If the base, A E, of an equilateral triangle (Fig. 489) represents to any scale a distance of 1,000, and if differences in elevation for this distance and for different angles of elevation be laid off on A B with a unit ten times as large as that used upon A E, for an angle of 5° 46′, the difference of elevation for a distance of 1,000 is 100, and the point will fall at B. Radial lines are drawn from E

* The values of $R \cos.^2 e$ are given within the same limits, the values e varying by intervals of from 1° to 20′.

through the points of division on A B, and are marked with the vertical angle to which the points correspond. Divide A B and A E decimally, and draw two systems of parallel lines through these points of division, one system parallel to A B and the other paral-

FIG. 489.



lel to A E. To determine the difference in elevation corresponding to any given distance and vertical angle, follow the line parallel to A B, through the point on A E corresponding to the distance, to its intersection with the radial line corresponding to the vertical angle. Transfer this point to A B by the nearest line of the system parallel to A E, and read the difference in elevation from the scale of heights A B, divided decimally from A to B.

To avoid the very oblique intersections that would obtain if the side A B were continued straight, and the differences in elevation for large angles were laid off upon it, the direction of the lines transferring distances is changed twice.

In the complete diagram the lines transferring distances are in each triangle parallel to the outer side: in A B E parallel to A B, in B C E parallel to B C, and in C D E parallel to C D. The scale of distances may be repeated on E D to avoid transferring through two triangles to get points of intersection with radial lines corresponding to vertical angles between 11° 48′ and 18° 26′.

Lines transferring elevations (shown broken in the figure) are

parallel to A E in the first triangle, parallel to B B′ in the second (B′ is the middle point of E C), and parallel to B′ B″ or C C′ in the third triangle (the points B″ and C′ divide E D into three equal parts). The line B B′ B″ transfers elevations of 100, C C′ transfers elevations of 200, and the point D corresponds to an elevation of 300. Therefore B C is to be divided decimally, and marked with elevations 100–200.

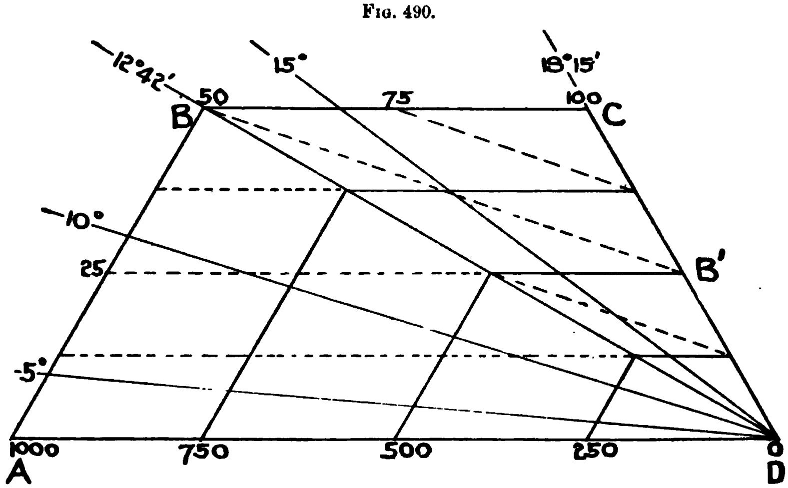
C D is divided decimally, and marked 200–300. E B″–B″ C′ and C′ D may be divided decimally, and marked 0–100, 100–200, and 200–300 respectively, in order to save transferring around to A B and B C from the third triangle.

A convenient sized diagram is obtained by making A E = 10″. Lines parallel to A E, A B, B C, and C D are drawn 0·1 inch apart, every tenth line heavy. To avoid confusing the diagram, only every alternate line of those parallel to A E is continued parallel to B B′ and B′ B″.

Draw radial lines for ten-minute intervals. Make the degree lines heavier than for intermediate angles.

The diagram will be clearer if different colored inks are used for

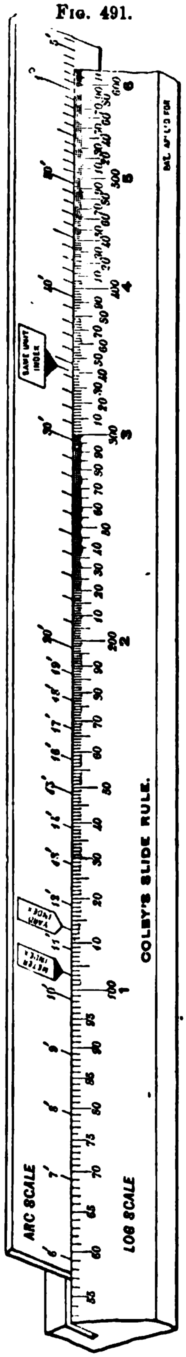
Fig. 490.



the three sets of lines and numbers: *e. g.*, distances, black; angles, red; elevations or corrections, blue.

For correction on distances (Fig. 490), let the entire base A D

FIG. 491.



represent 1,000; let the distance A B represent a correction of 50 on the distance A D, which will correspond to an angle of 12° 55′ (approx.); C will correspond to an angle of 18° 26′ (approx.), the correction being 100. Lines to transfer distances are parallel to A B and B C, and corrections on the distance are transferred parallel to A D and B B′, B′ being in the center of C D″. For this diagram intervals of 1° for small angles, decreasing to 20′, will be small enough.

The angles for which $\frac{1}{2}$ sin. 2 e is equal to 0·1, 0·2, 0·3, are respectively 5° 46′, 11° 48′, 18° 26′.

On account of the small variation in the cosine for small angles, instead of making a diagram to give the values of R $\cos.^2 e$, it is constructed to give the values of R $\sin.^2 e$, which is a small quantity, that changes rapidly. This correction, subtracted from R, gives the reduced distance:

$$R - R\sin.^2 e = R(1 - \sin.^2 e) = R\cos.^2 e$$

This diagram may be made of a trapezoidal outline changing the ratio of the two scales, and calling one end 50′, the entire base being 1,000′. The angle for which $\sin.^2 e = 0{\cdot}05$ is 12° 55′, and for which it equals 0·1 is 18° 26′.

Diagrams similar to the above have been called *self-reading* to distinguish them from the earlier forms, which required the use of a pair of dividers and a scale to take off the results.

787. Colby's Slide Rule. "The length of this slide rule (50 inches) makes it impossible to show a cut of the complete instrument. The cut shows about 17½ inches of the slide rule near the middle and reduced more than one half. The scales meet at an obtuse angle for convenient reading.

The arc scale (Fig. 491) slides easily in the groove, and will always work easily and never "pinch." The slide rule has three indexes, allowing distances to be read in feet, yards, or metres, as desired, thus meeting all requirements.

"*Directions for Using.*—Suppose the distance read between the two points is 340 feet (Fig. 491), and the vertical angle is 30′; slide the arc scale until the same unit index is opposite 340, the given distance; then upon the logarithmic scale, at a point opposite 30′, the given vertical angle, read 2·97 feet, the difference of elevation sought. If the vertical angle were 1°, the difference in elevation for 340 feet would be 5·93 feet. This simple operation of setting an index opposite a number corresponding to a distance, and then reading a number opposite a given graduation of arc, is all that is necessary in using this slide rule."

788. The Wagner-Fennel Tacheometer is one of a type of instruments used to some extent in both France and Germany, but not in this country. It is like a theodolite, but has no vertical circle, having instead a mechanical attachment enabling one to read from one of two scales the product of any distance parallel to the axis of the telescope by the sine of its angle of inclination, and from another the product of the same inclined distance by the cosine of the vertical angle.

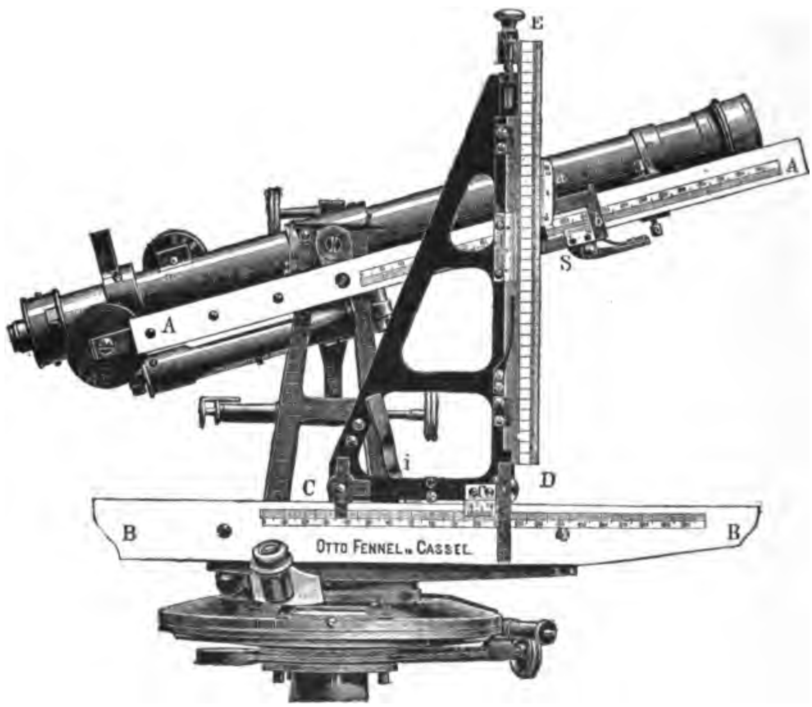
Description.—A piece A A (Fig. 492), supporting a scale is fixed by two arms to the telescope. One of these is attached to the end of the axis of the telescope, and the other is near the objective. The upper edge of this scale is parallel to the line of collimation. A slide, S, provided with two verniers, is held in position at any point on A A by light contact springs. The upper vernier (*a*) serves to read on the scale of heights D E when the edge of the scale is placed in contact with it. To make this contact with D E for all inclinations, the vernier pivots about a point placed exactly in the edge of the scale A A.

Almost directly below the scale A A, lying in a parallel plane and fixed parallel to the horizontal limb of the instrument, is another scale B B.

A triangular frame, or *projection apparatus*, supporting a vertical

scale E D, and a vernier c, rolls upon B B. After sighting to the rod, the frame is moved until vernier b coincides on the scale A A with the rod reading R. If the rod were held normal to the line of sight the reading R would be the inclined distance, and R cos. e

Fig. 492.



read at the vernier c would be the corrected horizontal distance, and R sin. e read between the zeros of the verniers a and d would give the difference in elevation.

It is not as convenient to hold the rod normal to the line of sight as to hold it vertically. If, however, the reading on B B ($=$ R cos. e), obtained as above, be set off upon the scale A A, and the verniers a and c read, the former will give the value R cos. e sin. e, and the latter R $\cos.^2 e$, which are the terms desired.

When the zero of vernier b coincides with the zero of the scale A A, vernier c reads zero. The scale D E has an adjustment in the direction of its length. It is divided decimally, but no numbers

are engraved upon it. An ivory strip permits the marking of the division temporarily with any desired values. If they are so marked that the elevation of the station occupied may be set opposite the zero of vernier *d* by the adjustment at E when vernier *b* reads zero, then, for a given value of R laid off on A A the elevation of the point sighted to will be read from the vertical scale D E. The above instrument used in connection with a plane table is called the *tacheographometer.**

The weight and complexity of the instrument, and the liability of so many movable parts getting out of adjustment, are the objections to this instrument.

It is a question whether the verniers can be set in the field as quickly as notes can be reduced with the tables and diagrams at hand.

789. The Plane Table. While the principle of the stadia is now universally employed for making locations, whether it should be used in connection with the transit or plane table is a question upon which topographers disagree. Both methods are used in this country by the Coast and Geodetic and the Geological Survey.

The points in favor of the plane table are: Economy, since the map is made at once without the expense of notes and sketches; and, as the mapping is all done upon the ground to be represented, all its peculiarities and characteristics can be correctly reproduced. A great advantage, also, is its capability of locating with reference to three visible known but inaccessible points.

On the other hand, the plane table is an instrument useful only for taking topography; the rodmen are idle while the mapping is being done; the instrument (for the same accuracy) is more unwieldy than the transit on difficult ground; the record of the work of a long period is constantly exposed to accident; the distortion of the paper with the varying dampness of air introduces errors in the map; while the area exposed makes it too unstable to use in high winds.

It should not be adopted as the instrument for a large survey

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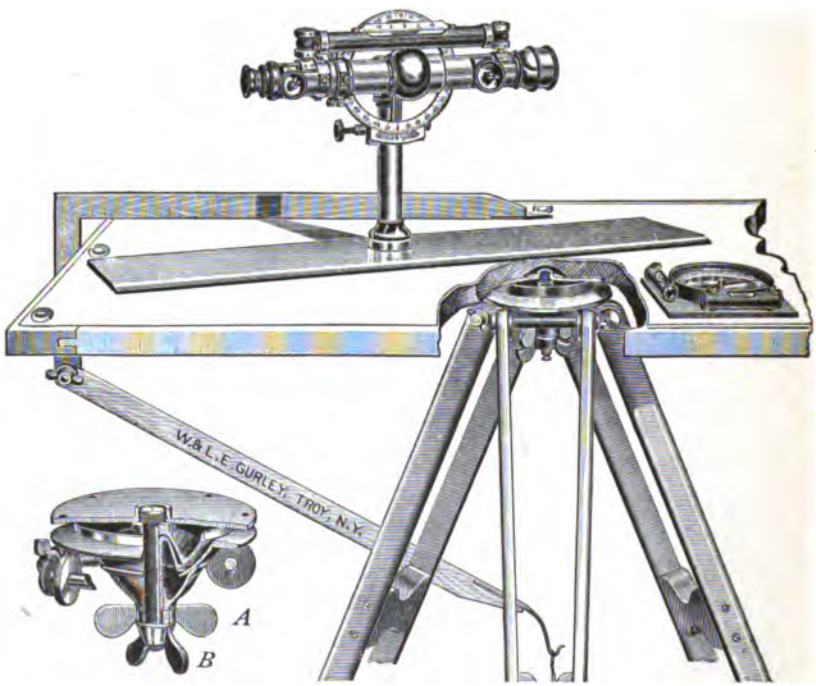
* For a complete description, see "Les Tacheomèters Wagner-Fennel," Otto Fennel. Paris: Gauthier-Villars.

unless the climate will permit continuous work under the above limitations.

790. Instruments. It is usually a rectangular board of well-seasoned pine, arranged in sections so as to prevent warping, about twenty inches wide and thirty long. The paper to be drawn upon may be attached to it by thumb tacks, or by clamping plates fixed on its sides for that purpose, or by springs pressed upon it, or it may be held between rollers at opposite sides of the table. Tinted paper is less dazzling in the sun than white.

The usual method of connection between the tripod and the board

FIG. 493.



is by means of a ball-and-socket joint capable of being clamped in position when the table has been leveled and oriented.

The plane-table movement shown in Fig. 493 is an improved form that has been adopted by the United States Geological Survey.

The improvement provides a means whereby the table, after

having been leveled up and clamped by means of the wing nut (A), may be turned in azimuth about the vertical axis shown by unscrewing the wing nut (B).

A tangent screw is also attached for accurately orienting the board.

A detached level is placed on the board to test its horizontality; though a smooth ball, as a marble, will answer the same purpose approximately.

A pair of sights, like those of the compass, are sometimes placed under the board, serving, like a "watch telescope," to detect any movement of the instrument.

For placing a point on the board exactly over the corresponding point on the ground, which is sometimes necessary, a plumbing arm is needed. The one shown in the figure has the end of the arm resting upon the paper brought to a sharp point by means of a metal tip; the lower arm is hinged, having a wing nut for fixing it at the proper angle, so that when the upper arm is horizontal the hook on the lower arm is directly beneath the point on the upper; an index at the hinged joint gives the correct angle. A compass is sometimes attached to the table, or a detached compass consisting of a needle in a narrow box (called a declinator) is placed upon it, as desired. In the figure, the compass, which has a full circle, is placed upon a square brass plate, and by applying it to the edge of the ruler in any position the magnetic bearing of the edge of the ruler may be determined. The edges of the table are sometimes divided into degrees, like the "drawing-board protractor." It then becomes a sort of goniometer.

The Alidade.—The old-style alidade consists of a brass ruler about twenty inches in length, having two slotted sights like those of the ordinary compass, the beveled edge of the ruler being in line with the slots in the sights.

The modern alidade (see Fig. 493) consists of a telescope pivoted on a single standard fixed perpendicular to the plane of the ruler. It has a vertical circle, with adjustable vernier and an accurate level attached to the upper side of the telescope. It has a vertical tangent motion and stadia wires.

The line of sight of the telescope need not lie in the plane of

the beveled edge of the ruler; so, for stability, it is usually set back from the edge a short distance.

The line of sight and edge of the ruler need not even be parallel to each other, provided the horizontal angle between the two is fixed.

THE ADJUSTMENTS.

The edge of the ruler should be a straight line. To test it, draw a line along the edge, reverse the rule end for end, place the edge upon the line, and again draw a line. If the two lines coincide the edge is straight. An exception to this obtains: When one end of the rule is concave the same amount, the other end is convex. If this defect exists, sliding the rule endwise and drawing a line will detect it.

The Sights.—The two sights in the old-style alidade should lie in a plane perpendicular to its base, otherwise the same direction would not be given by sighting through the top as would be obtained by sighting through the bottom. This adjustment may be tested with a try square.

The Board.—1. The top should be a plane surface. Test it with a straight edge. 2. The top should be perpendicular to the vertical axis of the movement. To test, place on the board an accurate level; level the board by bringing the bubble to the center, reverse the table, and if the bubble has moved, correct one half by inserting washers between the board and its connection with the plane-table movement.

Levels on the Alidade.—Their testing and adjustment are the same as for a transit.

Telescope.—The line of sight should be perpendicular to the horizontal axis (adjustment, see Art. 323, Part I).

The horizontal axis should be parallel to the top of the table, same as adjustment, Art. 324, Part I.

The line of collimation may be made to coincide with the edge of the rule by setting two needles in the board in range with a rod ten feet away, then bring the edge of the rule in contact with the needles, and adjust telescope standard until the cross hairs intersect on the rod.

Vernier on Vertical Arc.—The vernier of the vertical arc should read 0 when the line of sight is horizontal. This is the same adjustment as Art. 328, Part I.

Level on Telescope.—The bubble should be in the middle when the line of sight is horizontal. This is the same adjustment given in Art. 327, Part I.

The Paper.—The United States Coast Survey* determined that strips cut longitudinally from drawing paper varied from ten to twenty-five per cent more than strips cut transversely from the same paper. This unequal expansion would be a source of error. The United States Geological Survey, on their work to eliminate this error, employed two sheets of paragon paper mounted with the grains at right angles to one another and with cloth between them. Tests of paper so prepared showed practically no difference in expansion.

Construction of the Projection for a Topographic Sheet for Plane Table Work.—In order that all distances may check on the plane-table sheet, it is best to plot the parallels of latitude and meridians upon the sheet in such a way that we may represent the actual effect of the curvature, or, in other words, wrap the sheet about the globe, allowing for the difference in scale. For the small areas treated on a topographic sheet we may consider the sheet as the development of the conic surface which is tangent to the middle parallel of the region covered by the sheet. Or we may treat it as the development of two conic surfaces, one for the upper part and the other for the lower; this is the usual method. Tables have been computed giving the length of a minute or second of longitude, and latitude for each minute of latitude from the equator to the pole, and also the ordinates of deviation from a straight line of the parallels and meridians when developed on the conic surface. Such tables may be found in the "Coast and Geodetic Survey Report" for 1884. They are based upon the elements of the Clark spheroid of 1866.

Before constructing the projection for a sheet we must determine its limits. We usually have a rough sketch of the topography

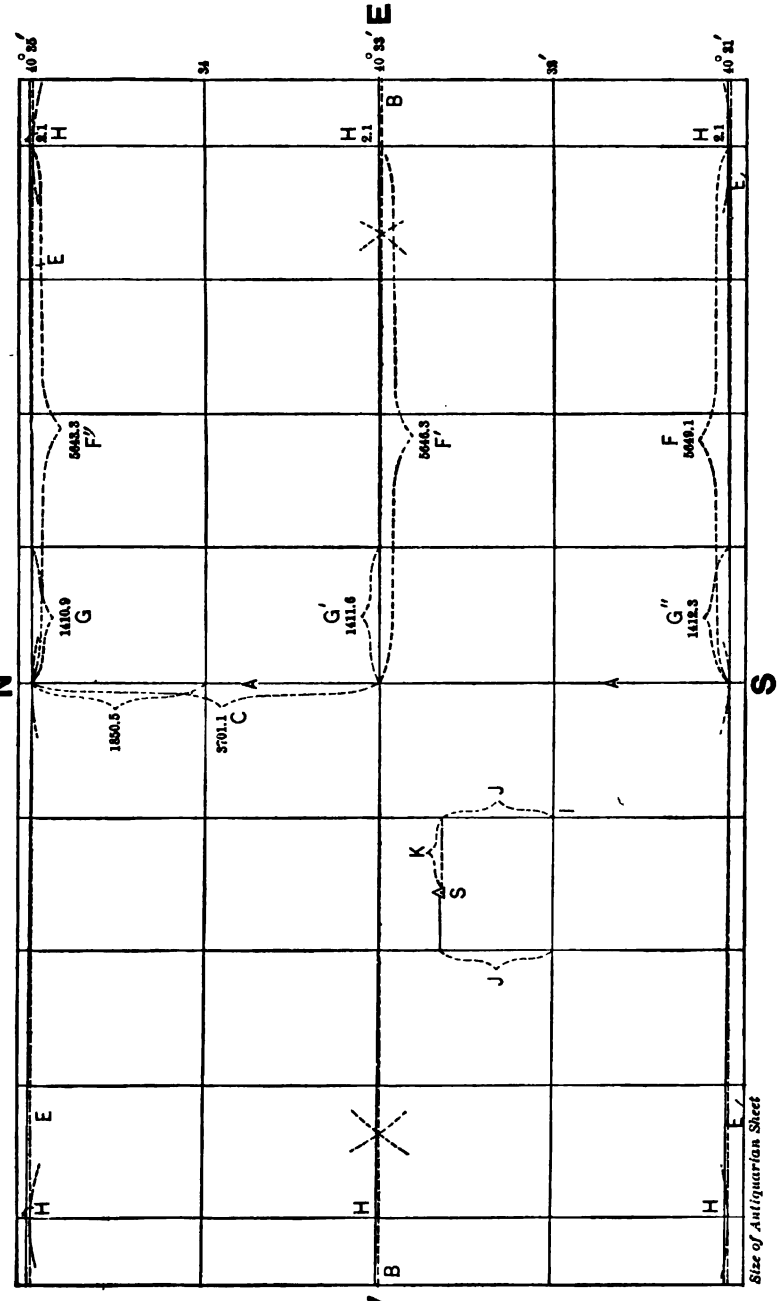
* "United States Coast Survey Report," 1862, p. 255.

we wish to cover by the sheet, and also the positions of the points of the triangulation in the region where the topography is desired. From these we can determine what parallel and meridian to put in the middle of the sheet.

Fig. 494 represents a topographic projection reduced to one eighth its natural size. It is made in the following manner: A A, the middle meridian, is a straight line drawn in the center of the sheet, and at its central point a perpendicular is very carefully constructed. A straight edge, fine-pointed beam compass, and an accurate scale are necessary for this kind of work. From the tables we take the length (in this case) of 2′ of latitude (or "meridional arcs," as they are called in the table) for the latitude of the sheet —viz., 40° 33′. Taking this distance in the beam compass we describe arcs each side of the perpendicular B B, at its middle point and also near its extremities. We then draw the straight lines E E, E_1 E_1, which are parallel to B B.

We next take the length of 4′ (in this case always use the greatest number possible) of longitude obtained from the tables giving "length of arcs of parallels," and lay off the distances H from A A. We must be careful, however, to take the distance for the same parallel upon which we lay it off. The lower parallel is longer than the upper, it will be noticed, owing to the convergence of the meridians. It is usually sufficient to lay off the minutes of longitude upon but three parallels, as shown in the figure. The meridians are then constructed by joining the points on B B with those on E E and also on E_1 E_1. If, however, the projection is large, and on a small scale, it may be necessary to consider the deviation of the meridians from a straight line, or lay off the ordinate X given in the tables, but this is not necessary in ordinary work. The ordinate Y for the deviation of the parallels must be considered, however, as it is very appreciable, as shown in the figure. For the ordinary projections we may take the ordinate for the end meridians and make the parallel a straight line between the end and middle meridians. Actually it should be a curve, of course, but the scale is too small to show it. The dotted horizontal lines show the perpendiculars to A A from which the ordinate Y must be laid off. The parallel must curve toward the pole—i. e.,

FIG. 494.


N
S
E
W
40° 35′
34
40° 33′
32′
40° 31′
5643.3
5646.3
5649.1
1410.9
1411.6
1412.3
1850.5
3701.1
Size of Antiquarian Sheet

the ordinates must be laid off above the horizontal lines. Parallels 32′ and 34′ are drawn midway between 31′ and 33′ and 33′ and 35′ respectively, and the intermediate meridians by joining the points on B B with those on E E and also on E_1 E_1, which are obtained by dividing H H on each into the number of equal parts required to represent the minutes of longitude. The meridians and parallels are then inked with a very fine ruling pen, and the projection is completed.

Before taking it into the field, however, all points whose positions are known are plotted upon the sheet. For this purpose we find the distance corresponding to the seconds of the latitude and longitude of each station from our projection tables. Suppose we have the position of the point s given—viz., $\phi = 42° 32' x''$ ($x'' = J$ in metres), and longitude such that the seconds $= k$ metres. We lay off J from the 32′ parallel on the meridian each side of the station, and join these two points with a fine pencil line, then lay off k on this line. Each station is plotted in this way.

As a final check upon the work, the diagonals of the small rectangles (practically) should be equal, and the distances between the triangulation points should be the same as used in determining the geodetic positions of the stations.

The sheet may now be taken into the field and a final check upon both the projection and the computation obtained by orienting the sheet at several of the triangulation stations and cutting upon all the stations visible from each. If the cuts do not intersect on the plotted position of the station an error may be looked for.

791. Organization of Party. A party for the taking of topography, using the plane table, is much the same as with a stadia; however, on account of the weight of the instrument—tripod head, board, etc.—means of transportation must be employed.

A less number of rodmen can be employed than with stadia, owing to the time required for mapping.

An observer, a man to reduce stadia notes and sketch topography around points determined by intersection or stadia from the plane-table station, and one rodman, will make the minimum work-

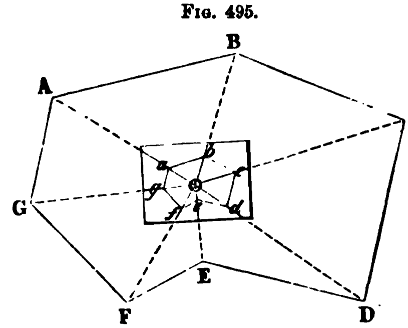
ing party, in addition to which axemen, and a team for transportation will usually be required.

792. Methods of Work. There are four methods of locating points, using a plane table. They are known by the following names: (1) Radiation, (2) traversing or progression, (3) intersection, and (4) resection.

All of these methods are used in the course of the work of a topographical survey by use of the plane table, depending upon the accessibility and distance of the point to be located and the relation it bears to the station occupied.

Method of Radiation.—This is the simplest though not the best method of surveying with the plane table. It is especially applicable to surveying a field, as in Fig. 495. In it and the following figures the size of the table is much exaggerated. Set the instrument at any convenient point, as O; level it, and fix a needle (having a head of sealing wax) in the board to represent the station. Direct the alidade to any corner of the field, as A, the fiducial edge of the ruler touching the needle, and draw an indefinite line by it. Measure O A, and set off the distance, to any desired scale, from the needle point, along the line just drawn, to *a*. The line O A is thus plotted on the paper of the table as soon as determined in the field. Determine and plot in the same way O B, O C, etc., to *b*, *c*, etc. Join *a b*, *b c*, etc., and a complete plot of the field is obtained. Trees, houses, hills, bends of rivers, etc., may be determined in the same manner. The corresponding method with the compass or transit has been described. The table may be set at one of the angles of the field, if more convenient. If the alidade has a telescope, the method of measuring distances with a stadia may be here applied with great advantage.

FIG. 495.

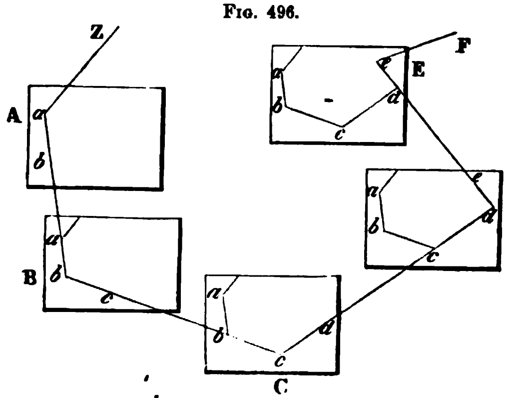


It is to be noted that this method gives no check upon the accu-

racy of the work. If, after locating all the points, a second sight is taken to the starting point, and the edge of the ruler coincides with the line first drawn, it is probable that the board has not moved during the work.

Traversing, or the Method of Progression.—Let A B C D, etc. (Fig. 496) be the line to be surveyed. Fix a needle at a convenient point of the plane table, near a corner, so as to leave room for the

FIG. 496.



plot, and set up the table at B, the second angle of the line, so that the needle, whose point represents B, and which should be named b, shall be exactly over that station. Sight to A, pressing the fiducial edge of the ruler against the needle, and draw a line by it. Measure B A, and set off its length, to the desired scale, on the line just drawn, from b to a point a, representing A. Then sight to C, draw an indefinite line by the ruler, and on it set off the length of B C from b to c. Fix the needle at c. Set up at C, the point c being over this station, and make the line $c\ b$ of the plot coincide in direction with C B on the ground, by placing the edge of the ruler on $c\ b$, and turning the table till the sights point to B. The plane table is then in position. The compass, if the table have one, will facilitate this. Then sight forward from C to D, and fix C D, $c\ d$ on the plot, as $b\ c$ was fixed. Set up at D, make $d\ c$ coincide with D C, and proceed as before. The figure shows the lines drawn at each successive station. The table drawn at A shows how the survey might be commenced there.

In going around a field, the work would be proved by the last line "closing" at the starting point; and, during the progress of the survey, by any direction, as from C to A on the ground, coinciding with the corresponding line, *c a*, on the plot.

This method is substantially the same as the method of surveying a line with the transit. It requires all the points to be accessible. It is especially suited to the survey of a road, a brook, a winding path through woods, etc. The offsets required may often be sketched in by the eye with sufficient precision.

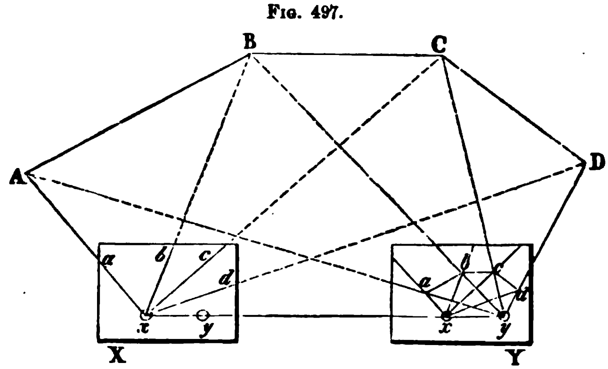
Lines run by this method between primary triangulation points, and called *traverses*, were used extensively by the United States Geological Survey for the control of their topography.

This is the best method of working, as it gives a complete check upon the work.

When the paper is filled, put on a new sheet, and begin by fixing on it two points, such as C and D, which were on the former sheet, and from them proceed as before. The sheets can afterward be united, so that all the points on both shall be in their true relative positions.

Method of Intersection.—This is the most usual and the most rapid method of using the plane table. Set up the instrument at

FIG. 497.



any convenient point, as X in Fig. 497, and sight to all the desired points, A, B, C, etc., which are visible, and draw indefinite lines in their directions. Measure any line X Y, Y being one of the points sighted to, and set off this line on the paper to any scale.

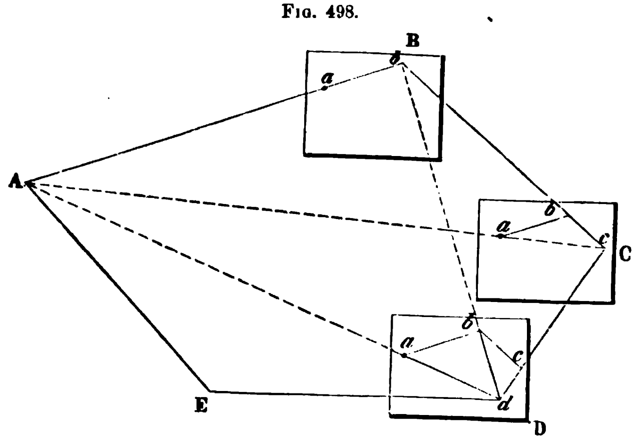
Set up at Y, and turn the table till the line X Y on the paper lies in the direction of X Y on the ground, as at C in the last method. Sight to all the former points and draw lines in their directions, then the intersections of the two lines of sight to each point will determine it. Points on the other side of the line X Y could be determined at the same time. In surveying a field, one side of it may be taken for the base X Y. Very acute or obtuse intersections should be avoided; 30° and 150° should be the *extreme* limits. The impossibility of always doing this renders this method often deficient in precision.

It is the only method for mapping inaccessible points, and before the introduction of the stadia was the method most employed in ordinary surveys.

With a great many points to intersect from a station, it is necessary to designate the point to which the different lines are drawn from the first station; to do this, place the name of the point in a rectangle, one side of which coincides with the line.

Method of Resection.—This method (called by the French *recoupement*) is a modification of the preceding method of intersection. It requires the measurement of only one distance, but all

FIG. 498.

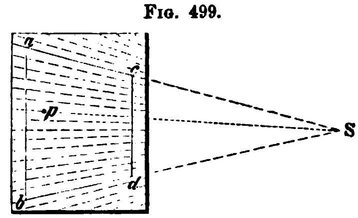


the points must be accessible. Let A B (Fig. 498) be the measured distance. Lay it off on the paper as $a\ b$. Set the table up at B, and turn it till the line $b\ a$ on the paper coincides with B A on the

ground, as in the Method of Progression. Then sight to C, and draw an indefinite line by the ruler. Set up at C, and turn the line last drawn so as to point to B. The table is then in position. Fix a needle at *a* on the table, place the alidade against the needle and turn it till it sights to A. Then the point in which the edge of the ruler cuts the line drawn from B will be the point *c* on the table. Next sight to D, and draw an indefinite line. Set up at D, and make the line last drawn point to C. Then fix the needle at *a* or *b*, and by the alidade, as at the last station, get a new line back from either of them, to cut the last-drawn line at a point which will be *d*. So proceed as far as desired.

*To Orient the Table.**—The operation of orientation consists in placing the table at any point so that its lines shall have the same directions as when it was at previous stations in the same survey.

Fig. 499.



With a compass this is very easily effected by turning the table till the needle of the attached compass, or that of the declinator, placed in a fixed position, points to the same degree as when at the previous station.

Without a compass the table is oriented, when set at one end of a line previously determined, by sighting back on this line, as at C in the Method of Progression.

To orient the table when at a station unconnected with others is more difficult. It may be effected thus: Let *a b* (Fig. 499) on the table represent a line A B on the ground. Set up at A, make *a b* coincide with A B, and draw a line from *a* directed toward a steeple or other conspicuous object, as S. Do the same at B. Draw a line *c d* parallel to *a b* and intercepted between *a* S and *b* S. Divide *a b* and *c d* into the same number of equal parts. The table is then prepared. Now let there be a station, P, *p*, on the table, at which the table is to be oriented. Set the table so that *p*

* The French phrase to "*orient* one's self," meaning to determine one's position, usually with respect to the four quarters of the heavens, of which the orient is the leading one, well deserves naturalization in our language.

is over P, apply the edge of the ruler to p, and turn it till this edge cuts $c\,d$ in the division corresponding to that in which it cuts $a\,b$. Then turn the table till the sights point to S, and the table will be oriented.

A great advantage of the plane table lies in the capability of setting up the instrument at an unknown point from which three known points are visible, and determining its position. This is done by the solution of the "three-point problem."

Four methods may be employed to determine the point:

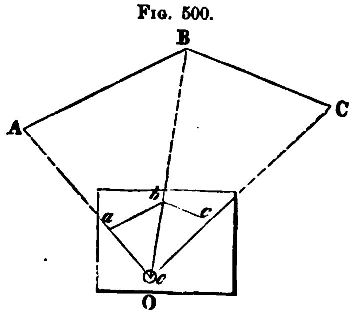
1. *A Mechanical Solution.*—Fasten a piece of tracing cloth or paper to the board, marking upon it a point to represent the unknown point. Draw through it lines toward the three known points. Shift the tracing paper until each of the three lines passes through the point on the paper corresponding to the point toward which it was drawn.

The position of the unknown station will be at the intersection of these lines. This is a rough method, not suitable for accurate work.

2. *By Intersection.*—If means are at hand for orienting the table, as given above, the point may be located by intersection, as follows:

Set up the table over the station, O, in Fig. 500, whose place on the plot already on the table is desired, and *orient* it by one of the means described above. Make the edge of the ruler pass through some point, a, on the table, and turn it till the sights point to the corresponding station, A on the ground. Draw a line by the ruler. The desired point is somewhere in this line. Make the ruler pass through another point, b, on the table, and make the sights point to B on the ground. Draw a second line, and its intersection with the first will be the point desired. Using C in the same way would give a third line to prove the work. This operation may be used as a new method of surveying with the plane

FIG. 500.

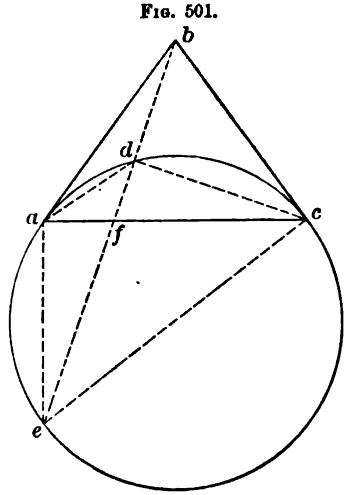


table, since any number of points can have their places fixed in the same manner.

3. *Geometrical Solutions.*—The position of the station may be determined by resection.*

In Fig. 501, let *a b c* be the points on the sheet representing the signals A B C on the ground. The table is set up and leveled at the point D, whose position on the sheet, *d*, is to be determined. The alidade is set upon the line *c a*, and, by revolving the table sight upon A and clamp, *a* being toward A, from *c*; then, with the alidade centering on *c*, the middle signal B is sighted, and the line *c e* drawn along the edge of the rule. Set the alidade upon the line *a c*, unclamp, and, by revolving the table sight to signal C, clamp again. Then, with the alidade centering on *a*, sight to the middle signal B, and draw *a e* along the edge of the rule. The point *e* (the intersection of these two lines) will be in the line passing through the middle point and the point sought. Set the alidade upon the line *b e*, direct *b* to the signal B by revolving the table, and the table will then be in position. Clamp it, centering the alidade upon a sight to A; draw *a d* along the rule. This will intersect *b e* in the point sought. To verify its position, centering the alidade on *c*, sight to C.

FIG. 501.



Demonstration: The opposite angles of the quadrilateral *a d c e* being supplementary, the angles *a c e* and *a d e* are subtended by the same chord *a e*, and *c a e* and *c d e* are subtended by the same chord *c e*, and consequently the intersection of *a e* and *c e* at *e* must fall on the line *d b*. Or, the segments of two intersecting chords in a circle being reciprocally proportional, the triangles *a d f* and *c e f* are simi-

* Bessel's method by inscribed quadrilateral, see "United States Coast and Geodetic Survey Report," 1880, p. 181.

lar, as also the triangles $c d f$ and $a e f$; and therefore d, f, and e must be in a right line passing through b.

The construction in Fig. 501 shows the point D within the triangle A B C. The same notation and construction will apply when D is without the triangle A B C; B in every case designating the central point as seen from D.

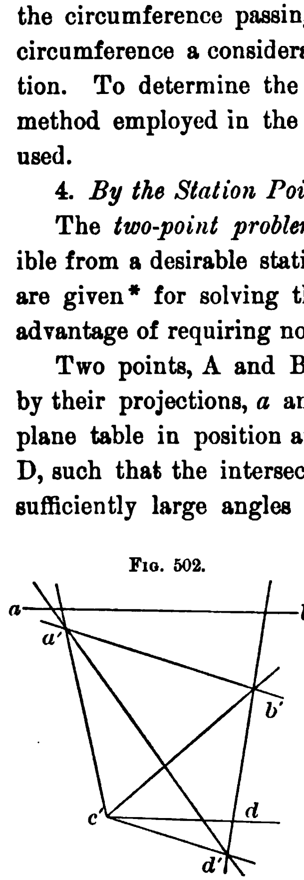
The position of the point d becomes indeterminate if D lies on the circumference passing through A B C; and if it lies near this circumference a considerable error is liable to be made in its location. To determine the location of the point D in this case, the method employed in the solution of the *two-point problem* may be used.

4. *By the Station Pointer.*—See Art. 817.

The *two-point problem*, if but two determined points are visible from a desirable station for the instrument. Several methods are given* for solving this problem; the one here given has the advantage of requiring no linear measurement.

Two points, A and B, not conveniently accessible, being given by their projections, a and b (Fig. 502), it is required to put the plane table in position at a third point, C. Select a fourth point, D, such that the intersections from C and D upon A and B make sufficiently large angles for good determinations. Put the table approximately in position at D, by estimation or by compass, and draw the lines A a, B b, intersecting in d'; through d' draw a line directed to C, and on this line lay off, from d', the estimated distance, C D, and mark the point thus found c'. Set the instrument on C with c' over the point, and sight to D, with the edge of the rule coinciding with $c'\,d'$. Draw lines from c' to A and to B. These lines will intersect the lines d' A and d' B at points a' and b', which form with c' and d' a quadrilateral similar to the true one, but erroneous in size (since the dis-

FIG. 502.



* "United States Coast and Geodetic Survey Report," 1880, pp. 184, 185.

tance $c' d'$ was assumed) and in position (since the table was not properly oriented at either station). The angle which the lines $a\ b$ and $a' b'$ make with each other is the error in position (orientation). By constructing now through c' a line $c' d$, making the same angle with $c' d'$ as that which $a' b'$ makes with $a\ b$, and directing the line $c' d$ to D, the table will be brought into position (or orientation), and the true point c can be found by the intersection of a A and B b.

Instead of transferring the angle of error by construction, we may proceed as follows: As the table now stands, $a' b'$ is parallel with A B, but it is desired to turn it so that $a\ b$ is parallel with A B. Place the alidade on $a' b'$ and set a mark in that direction; then place the alidade on $a\ b$, and turn the table until it again points to the mark; $a\ b$ will be parallel with A B, and the table in position.

Based upon the location of the true point with reference to the triangle of error, the locations of points with reference to the great triangle and great circle may be classified as follows:*

Class I. When the point falls within the great triangle the true point is within the triangle of error (at 1, Fig. 503).

To revolve the table into position: If the line from any one of the points falls to the right of the intersection of the other two, turn the table to the left; and if to the left, turn it to the right.

Class II. When the point sought falls (1st) within either of the three segments of the great circle formed by the sides of the great triangle as chords (at 2), or (2d) without the great circle and within the sector of the opposite angle of either angle of the great triangle (at 3), the true point is on the side of the line from the middle point opposite to the intersection of the lines from the other two points. This also includes the case where the three fixed points are in a straight line. In the first case, if the line from the middle point is to the right of the intersection of the other two, turn the table to the right; and if to the left, turn it to the left. In the second case, when the line from the right-hand station is uppermost, turn the table to the right; and when that from the left hand is uppermost, turn it to the left. The area covered by Class II is shown shaded in the figure.

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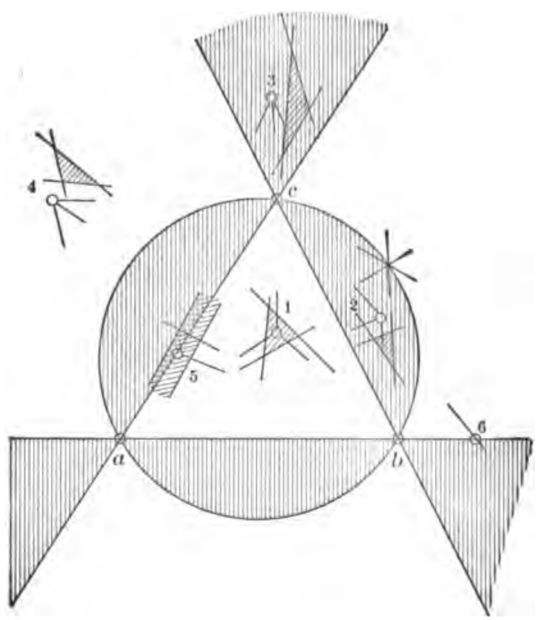
* "United States Coast and Geodetic Survey Report," 1880, p. 182.

Class III. When the point sought falls without the great circle and within the sector of either angle of the great triangle, the true point is on the same side of the line, from the middle point, as the intersection of the lines from the other two points (at 4).

To turn to position: If the line from the middle point is to the

FIG. 503.



right of the intersection of the other two, turn the table to the left; and if to the left, turn to the right.

In case the point sought falls on the range of any two of the points and the table is deflected from true position, the lines from the two points will be parallel, intersected by a line from the third point (at 5). This range can always be determined by alinement, the table set in position on the range, and the point occupied be determined by resection on the third point (at 6). When the line from the right-hand station is uppermost, turn the table to the right; and when that from the left is uppermost, turn it to the left.

Instead of turning the table to the left or right, according to the rules given above, a position of the true point may be assumed with reference to triangle of error, and a second approximation of

position for the table be made, this process being repeated until the station is located with the desired degree of accuracy.

The following rules are used in locating the true point: 1. The point sought is always on the same side of the line from the most distant point as the point of intersection of the other two lines. 2. At distances from the three lines drawn from the three fixed points proportional to the distances of the latter from the point occupied.

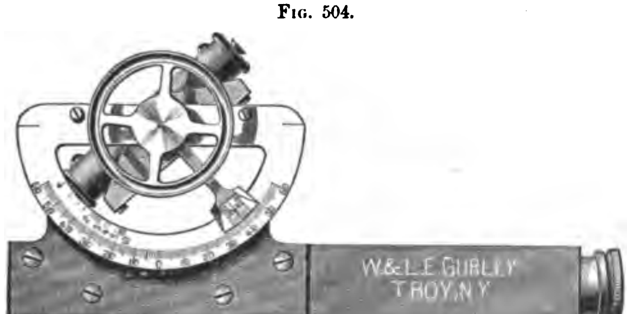
To make the first approximation to position as close as possible, a declinator should be used in orienting the plane table before constructing the triangle of error.

792. The Clinometer and **Hand Level** are used for the purpose of determining contours from points of known elevation, either to fill out portions of a stadia survey not easily accessible with the stadia, or parts whose topography can be obtained more easily, and with sufficient accuracy, without the stadia, or to take the topography for a distance on either side of a line determined by transit and level, as they are employed in railroad survey.

These instruments are also used without other means of control than the aneroid barometer for elevation and the prismatic compass for directions, with the use at times of a pedometer or odometer for measuring distances.

The instruments are the *Locke level*, described in Art. 508, Part I.

FIG. 504.



The *Abney hand level* and *clinometer* (Fig. 504) is a modification of the Locke level. The level tube is pivoted at the center of

a graduated arc. The bubble is seen by reflection in a mirror placed in the upper half of the square tube. When the bubble tube is level the bubble is seen through the tube to be bisected upon the edge of the mirror.

With a line of sight inclined at any angle, the bubble tube can be turned until the bubble is bisected as described, when the angle of inclination may be read from the graduated arc by means of the vernier arm fixed at right angles to the bubble tube. By setting the vernier reading at 0 the instrument may be used as a Locke level.

A scale of slopes is marked upon some instruments. The work could be facilitated by having the aperture of the level tube of a fixed diameter, or have two horizontal wires in it, so as to read the distance to the staff used, by the intercept upon the latter.

To work with the *Locke level* from points of known elevation, an assistant with a level rod is necessary for rapid and accurate work.

Set the target upon the rod at a distance above or below the height of the eye of the observer an amount equal to the difference in elevation between the station from which the observer reads and the contour next below or above that point. The assistant with the rod moves out in a direction at right angles to the line, if it concerns topography along a line, until the target as indicated by the level is at the same height as the eye. By pacing the distance from the station the contour may be located upon the sketch. As the rodman will have to determine the position by a tentative process, the observer should pace the distance. This point is then used as was the station; now, however, the target is moved either up or down one or two contour intervals from the point on the rod corresponding to the height of the eye of the observer, and succeeding contours located as before.

The observer could stand at the station and locate successive contours, within the limits of the rod, by setting the target at proper heights. While this would avoid cumulative errors in height, it is not as rapid nor as accurate in distances out as the other.

A single observer may sketch topography with the hand level by noting the point in which his line of sight intersects the slope on rising ground, then pacing the distance to such points. By

repeating this operation a number of times a height may be measured by a number of differences of level each equal to the height of the observer's eye. On a downward slope, the observer moves away from the point of known elevation until the hand level indicates that it is upon the same level with the eye.

Points near the line are located by offsets from the line; points at considerable distance, by taking their bearing from two different points on the line.

Distances may be paced accurately enough for topography taken with a hand level.

Used as a clinometer, the Abney hand level is employed to take the angle of inclination of the slope from the point of known elevation. For this purpose an assistant is required, having a rod with a target placed at the height of the observer's eye. The assistant holds the rod at a distance of about 200 feet from the line, in the case of a uniformly sloping surface, or at some point of change of slope, in which case the distance to the point is paced. Having the slope, a table of natural tangents will tell how far apart in plan the contours will be for a given contour interval. The following form of table is most convenient, giving the distance apart for various slopes of contours at one-foot intervals:

Degree of slope.	Distance apart of contours.	Degree of slope.	Distance apart of contours.	Degree of slope.	Distance apart of contours.
1°	57·3 feet	9°	6·31 feet	17°	3·2 feet
2	28·64	10	5·67	18	3
3	19·08	11	5·14	20	2·7
4	14·3	12	4·7	25	2
5	11·43	13	4·33	30	1·7
6	9·51	14	4·01	35	1·4
7	8·14	15	3·73	40	1·2
8	7·12	16	3·49		

From this same table the elevation of any point of change of slope may be obtained by dividing the distance to the point by the distance between contours for that slope. From the elevation of a point where the slope changes the inclination of the new slope may be read and contours spaced upon it.

The topography is sometimes taken by reading the slopes only,

and recording their inclination and distances to points of change of slope. It is, however, much better to put in the contours to their true elevation while in the field.

Topography is sketched either in books or on sheets. For topography over an area sheets are preferable, while for railroad work books quadrille ruled are ordinarily used, unless no other map is to be made than that made in the field, when sheets must be employed. In using sheets, points in common to two adjacent ones must be marked so that they may be properly joined.

Two methods are used for sketching topography along a line. One is, to use the center line of the page as the line surveyed, making a note of deflection to the right or left, but not plotting them. The points in favor of this method are, that it avoids the use of a protractor or scale in the field; the ruling of the page can always be used as a scale for distances parallel or perpendicular to the line. In the other method all deflections are plotted; this avoids the distortion of the topography at points where large deflections occur in the line. On this account this method is to be preferred where the ground is much broken.

In sketching on a preliminary survey, the topography should be a full day behind the line party, in order that all elevations for the next day's work can be copied the night before; or the two parties should keep together, so that no time shall be lost in obtaining elevations.

793^1. Photography has long been successfully employed by European engineers, notably those of Italy, for the purpose of taking topography. The Canadian Government has also employed it successfully in the survey of Alaska.

The recommendation of this method is the great saving of time in the field, while giving topographic features with all the accuracy required for maps to be plotted on a scale of 1 to 25,000.

M. Javary states that the maximum error both for horizontal distances and elevations, using a camera with a focal length of twenty inches and a microscope in examining the points, was only 1 in 5,000, as deduced from a number of cases.

M. Laussedat, in his work, found that this method did not require more than one third the time necessary by the usual methods.

This makes it especially suitable in all mountainous regions, where so much time is lost in getting to and from stations that but little is available for observations and sketching.

A single occupation of a station with photographic apparatus would suffice to complete work that with the ordinary methods would require several days.

Instruments.—The ordinary camera may be used, if it is provided with a level. A tripod head for leveling the instrument, and a roughly graduated horizontal circle for reading the direction of the line of sight, when photographing different parts of the horizon, are convenient attachments.

A camera is sometimes used upon a plane table, the record of the work being made upon the paper in connection with a set of radial lines drawn from the point representing the station occupied.

Many special forms of instrument combining the camera and theodolite have been devised, some one of which should be used if work of this kind is to be undertaken on a large scale. For a description of these instruments, and a complete treatise on this subject, comprising a discussion of the requirements of the apparatus, the fundamental principles of photography, methods of field work, forms of notes, reduction of notes and making of the map, together with the bibliography of the subject, see "United States Coast and Geodetic Survey Report," 1893, Part II, Appendix 3.

794. The Representation of the Configuration of the Surface of the Ground. There are two methods of representing the slopes and elevations of the earth's surface:

1. By contours.
2. By hill shading.

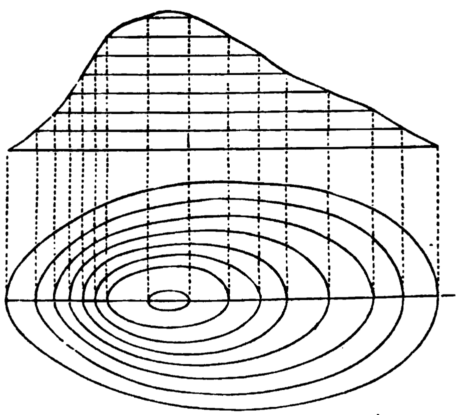
Although in the second method the contours do not remain as a part of the finished map, they must be put in in pencil as a basis of work for the hill shading.

In a mountainous country the method of hill shading produces more artistic results, and represents better to the eye the relative configuration of the surface; but it is not as convenient for taking off elevations as the method of contours. For this reason a combi-

nation of the two is sometimes employed to get the advantages of both methods.

795. Contours. The intersection of the surface of the ground by a horizontal plane is called a *contour*. The intersections of the surface by a series of equidistant parallel planes projected upon the map show the configuration of the surface (Fig. 505). The closer the contours the steeper the slope. If the planes be taken at some even distance apart —one, five, ten, or twenty feet, etc.—the elevation of any point upon the surface will be determined.

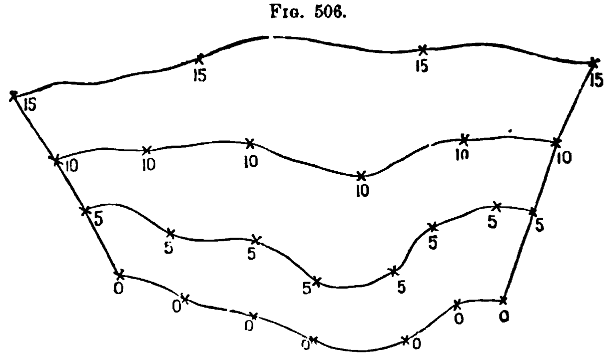
Fig. 505.



Shore lines of still water are an illustration of contours.

The horizontal planes whose intersections give the contours are spaced at regular intervals, called the "contour interval," beginning at the *plane of reference*, or *datum*. If points are accessible whose true elevation is known,

Fig. 506.



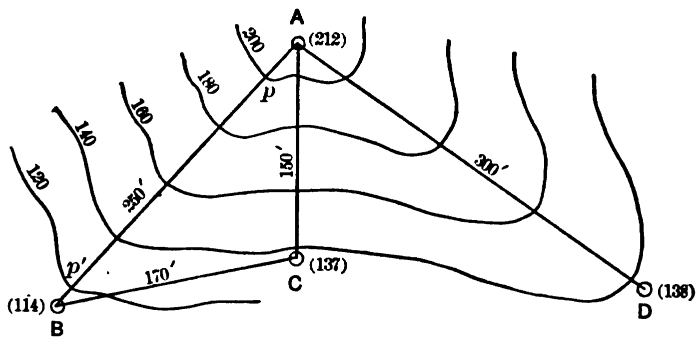
the benches of the survey should be determined from these points, in order that all points may be given in true elevation.

In assuming a *datum*, the aneroid may be used in estimating the

elevation of the bench from which to start; or, if there is no object in having elevations referred to sea level, the datum should be assumed lower than any point to be represented, so that no contours will have a negative elevation.

The Contour Interval.—The contour interval varies with the scale and the use of the map. For city and park work it is 1 or 2 feet; for preliminary railroad work, 5 or 10 feet; for United

FIG. 507.



States maps, scale 1 to 62,500, 5 to 50 feet; scale 1 to 125,000, 10 to 100 feet; scale 1 to 250,000, 200 to 250 feet.

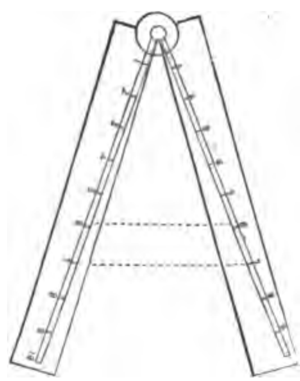
There are two general methods of drawing in the contours, depending upon the method of taking the topography. If points were located upon the contours only, these points are plotted and joined by a curved line following the undulations of the earth (Fig. 506). If points of change of slope were determined on the survey either by the *irregular* methods (Art. 776), or by the *regular* method of cross-sectioning, then between these points the slope may be assumed to be uniform, and points of intermediate elevation may be determined by interpolation (Fig. 507).

Having determined the location of a number of points of the elevation of the desired contour, these are joined by a line curving in and out, following the configuration of the surface.

A little practice will enable one to make these interpolations with sufficient accuracy without mechanical aid, but for this purpose a sector may be employed (Fig. 508). Open the sector until the distance between the two numbers on the arms corresponding to the difference in height of the two points is equal to the distance

between the two points upon the map. Then the distance from one of the points to any point of intermediate height will be equal to the distance between the two arms taken at numbers on the scales corresponding to the difference in elevation between the given point and the point desired.

FIG. 508.



The observance of the following points will be of aid in drawing the contours:

1. Slopes between given elevations are assumed to be uniform.

2. All points of the same elevation that can be joined by a line which neither crosses higher nor lower ground will have a continuous contour drawn through them.

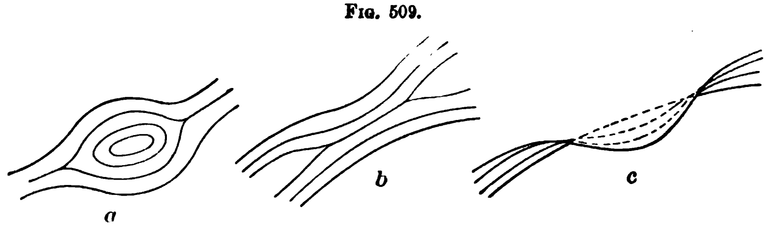
3. Contours are spaced at equal intervals upon a surface of uniform slope; if the surface is a plane, they are straight parallel lines.

4. A contour never splits, as shown in Fig. 509, at *a*, nor do two contours unite, as shown at *b*.

5. Contours do not cross over each other. The only exception to this is in the case of an overhanging cliff, shown at *c*, which is, however, of very rare occurrence.

6. A contour can not have an end within the drawing; it must

FIG. 509.



either close upon itself, or, if it enters at the edge of the map, it must terminate at some other point on the edge of the map.

7. The highest contours along ridges, and the lowest contours in valleys, go in pairs; because the lowest horizontal plane that would intersect a valley must intersect it in two lines, and the highest

horizontal plane that will intersect a ridge must intersect it in two lines.

8. At streams and ravines contours turn upstream, coinciding with the line defining the stream. If the bed of the stream or bot-

FIG 510.

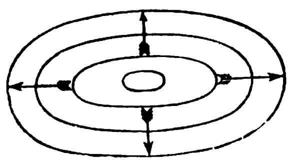
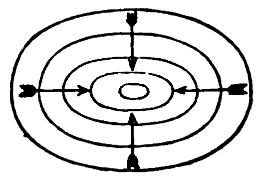


FIG. 511.



tom of the ravine rises above the plane of this contour, then the contour crosses it.

9. Contours are perpendicular to lines of steepest slope, and also to ridge and valley lines.

Familiarity with the meaning of the following arrangements of contours will facilitate the construction and interpretation of a map:

1. A closed contour, with one or more higher ones inclosed, is a hill (Fig. 510). The arrows show the direction in which water would run.

2. A closed contour, with one or more lower ones inclosed, is a hollow or depression (Fig. 511).

3. An area partially inclosed with contours, having the higher ones on the inside (the contours will have their concave side to-

FIG. 512.

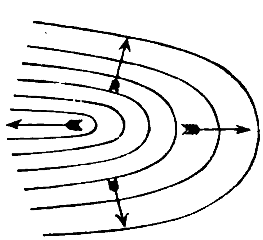


FIG. 513.

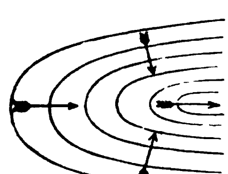
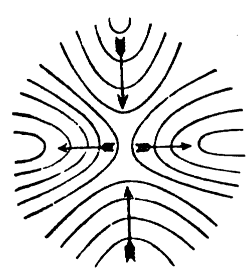


FIG. 514.



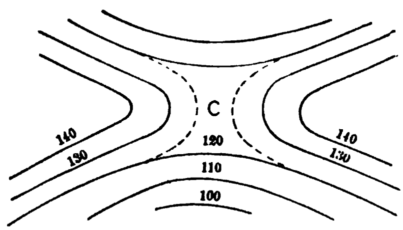
ward the higher ground), is a *croupe*, the end of a ridge or promontory (Fig. 512).

4. An area partially inclosed with contours, having the lower

ones on the inside (the contours will have their concave side toward the lower ground), is a valley or *thalweg* (Fig. 513).

5. Four sets of contours like Fig. 514, with their convex sides toward each other, represent a *col* or a *saddle.* A "pass" in a mountain range is a *col.* It is a low point in a ridge, and may be defined geometrically as the lowest point of the line of intersection of two slopes which rise above this intersection.

FIG. 515.



If valleys head in toward this col, there will be two other sets of contours like Fig. 513 between the first set, as shown in Fig. 514; otherwise the arrangement will be as shown in Fig. 515.

In taking the topography the elevations of all cols are of great importance, as indicating the limit of contours passing through them. If the col (C, Fig. 515) is intermediate in elevation between 120 and 130, the 120 contours do not pass through; if the col is below 120, they are drawn, as shown, with the broken line.

Ridges and Thalwegs.—The general character of the surface of a country is given by two sets of lines: the *ridge lines*, or *watershed lines*, and the *thalwegs*, or *lowest lines of valleys.*

The former are lines which divide the water falling upon them, and from which it passes off on contrary sides. They are the lines of least slope when looking along them from above downward, and they are the lines of greatest slope when looking from below upward. They can therefore be readily determined by the slope level, etc. They are the lines of *least* zenith distances when viewed from either direction.

On these lines are found all the projecting or protruding bends of the contour lines, convex toward the lower ground, as shown in Fig. 516.

The second set of lines, or the *thalwegs*, are the converse of the former. They are indicated by the water courses which follow them or occupy them. They are the lines of greatest slope when looked at from above, and of least slope when looked at from be-

low. They are the lines of *greatest* zenith distance when viewed from either direction.

On these lines are the receding or re-entering points of the contour curves, concave toward the lower ground.

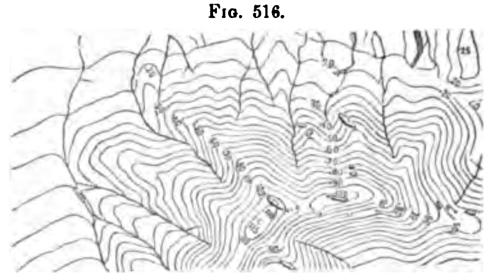
The general system of the surface of a country is most easily characterized by putting down these two sets of lines and marking the changes of slope, especially the beginning and the end.

In taking topography the most important points to be determined are:

1. At the top and bottom of slopes.
2. At the changes of slopes in degree.
3. On the watershed lines and on the *thalwegs*.
4. On cols, or culminating points of passes.

Contours.—Contour lines should be neither angular nor finely waved, but be drawn in smooth curves. On a long slope or hill, draw first the bottom contour line and the top one, then the middle one, and afterward interpolate others. Contours are inked in with red ink or crimson lake upon maps when the topography is represented by color. They are put in with burnt sienna if the topography is pen work in India ink. A common pen, with practice, gives the best results. They should not be drawn through rocks, nor buildings, nor across roads, streams, etc., except on maps to a very small scale. The *references* — numbers giving the elevation of the contour — are placed on the upper side of the contour with their bases resting on the contour, or in breaks left in the contour. These references are more prominent and look better if they follow a line normal to the contours.

Fig. 516.



References should be placed at sufficient intervals to enable elevations at any point to be quickly determined. Every fifth or tenth contour should be made heavier than the others.

If a summit or a depression can not be shown as accurately as desired with the contour interval used, an intermediate contour is drawn in a broken line to distinguish it from the others, and at the same time its elevation is marked.

796. Hill Shading. There are three different methods of representing topography by means of hill shading; they are

1. *The Horizontal System*, in which the surface is covered with strokes or hachures parallel to the contours.

2. *The Vertical System*, in which the surface is covered with hachures normal to the contours.

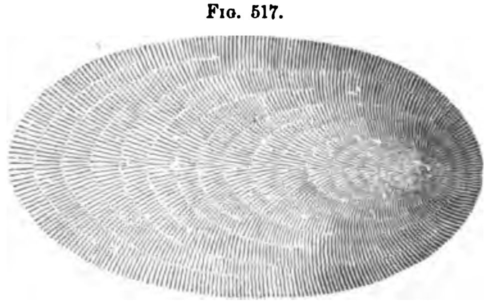
3. *The Brush System*, in which the surface is covered with a shade laid on with a brush.

Hill shading brings out better the relief of the surface than do contours. The great objections to its use are the time necessary for finishing the map; when used alone, the map does not give actual elevations; and all the details, especially on steep slopes, are greatly obscured.

In the above methods two systems of illumination are used, the vertical and the oblique.

Scales of Shade. Vertical Illumination.—The steeper the slope the less light it receives per unit of area and the darker it will appear.

FIG. 517.

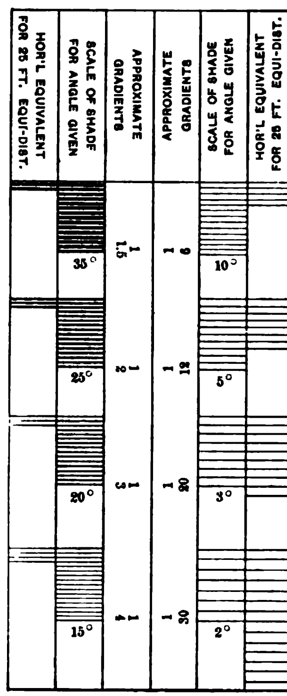


Horizontal surfaces are made white. The gradation of shade for varying slopes is determined by rules which greatly exaggerate the differences in shade, since in the methods employed the intensity varies directly with the angle from white at 0° to black at from 45° to 75°, depending on the method. Fig. 517 represents an oval hill by this system.

Oblique Illumination.—With oblique illumination the side next the light, which is considered as coming from the upper left-hand

corner of the map, making an angle of 45° with the vertical, will be lightest, while the side most remote from the light will appear darkest. The gradation in shade now used in this method, as differing from that due to vertical illumination, may be better understood by considering the illumination of a right cone with a circular base. With vertical illumination it would have a shade of uniform intensity; under the oblique illumination the element adjacent to the light would have a shade of an intensity represented by unity; the element most remote from the light would have a shade represented by an intensity of 4, while the two elements along which planes parallel to the light would be tangent would have a shade of an intensity of 2.

Fig. 518.



The work to be done preliminary to the use of any of the systems of hill shading is:

1. Put in contours in pencil.
2. Make a scale for measuring inclination of slopes.
3. Make a working scale.
4. Draw guide lines that the hachures must follow.

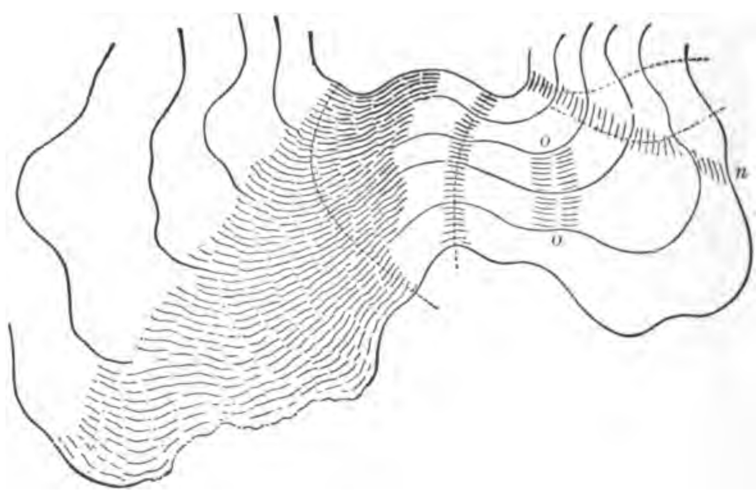
797. The Horizontal System. The *horizontal system* with vertical illumination is the system in use in England and is known as the English system. The scale of the map remaining the same, the number of hachures in a contour interval would increase with it, being twice as many for a 50-foot contour interval as for a 25-foot one. The following points are to be noted in the use of this system: Hachures are made from $\frac{1}{60}$ to $\frac{1}{2}$ inch in length, the longer ones being the finer; they follow the direction of the contours, being normal to lines of steepest slope.

Each hachure is of the same breadth throughout. They are

drawn in sets, beginning at the top and working down, blending the sets into each other to prevent white streaks down the slope, shown at *o o*, Fig. 519. They are drawn toward the draughtsman, and from left to right. If contours are retained in the finished map they are put in in red, or with a line that can be easily distinguished from the hachures.

Figs. 518 and 519 show the English working scale, and illustrate the manner of putting in the hachures. Fig. 520 shows a map on the horizontal system with oblique illumination.

FIG. 519.



798. The Vertical System with Vertical Illumination. The most important method under this system is known as *Lehmann's* or the *German method.* It will be described, as it is the basis of that employed by the United States Coast and Geodetic Survey and in Austria, and is similar to the French system.

The German or Lehmann's Method.—He uses nine grades for slopes from 0° to 45°, the first being white and the last black (Fig. 521). For the intermediate slopes he makes the white to the black in the following proportion:

The white : the black : : 45° — angle of slope : angle of slope.

For example, for 30°:

Light : shade : : 45° — 30° : 30° : : 1 : 2.

Hence, the space between the strokes is to their thickness as 45° minus the angle of the slope is to the angle of the slope. Slopes steeper than 45° are represented by hachures of a greater

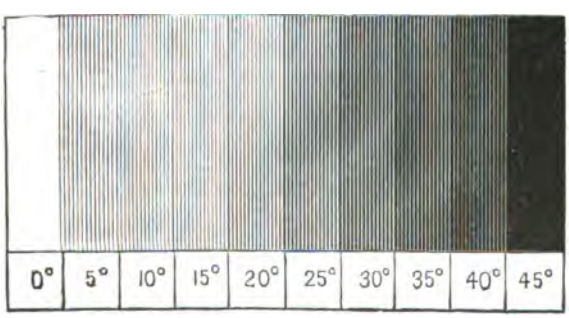
Fig. 520.

breadth than those for the 40° slope, but showing the same width of white between them.

Fig. 522 is a hill drawn by Lehmann's method.

Also, to distinguish slopes varying from 0° to 5°, an exception

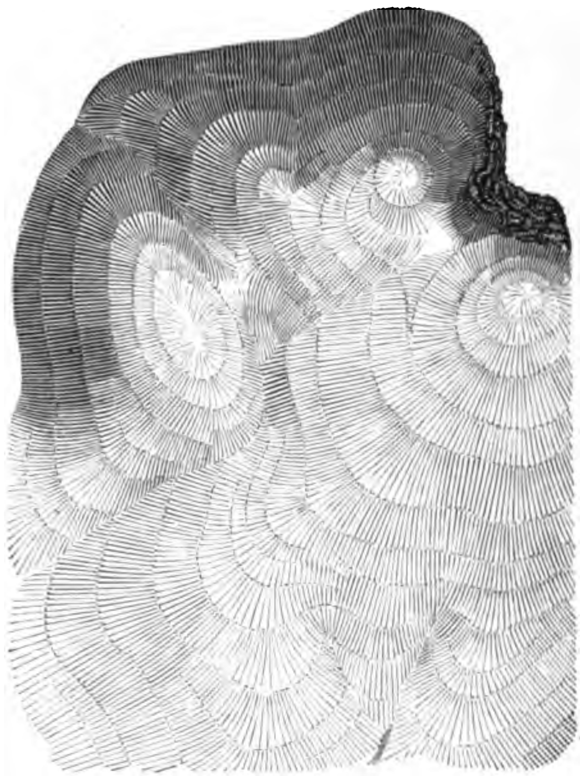
Fig. 521.



was made to the rule given above, and the same map space which for a 5° slope is covered with 5 hachures, has but 4 hachures for a

4° slope, 3 for a 3° slope, etc. In Lehmann's standard working scale the width of a hachure for a 5° slope is specified at one ninth of a millimetre. From this the width of hachures for other slopes can be determined from the proportion given.

FIG. 522.



Usually the number of hachures per inch will be determined by the scale of the drawing, 40 to 60 per inch being usually employed.

Fig. 523 shows a map on the vertical system with oblique illumination.

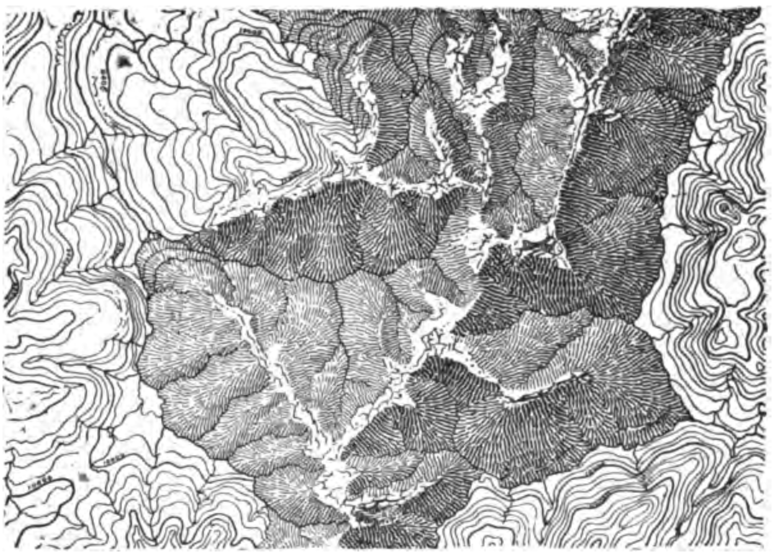
The Austrian Scale substitutes 50° for 45° in Lehmann's scale, and has ten grades of shade, using the same convention for slopes of 1° to 5°.

The United States Coast and Geodetic Survey Scale of Shade and Working Scale.—Lehmann's scale, from 5° to 25° inclusive,

was adopted; but from 25° up the proportion of black was increased less rapidly, in order to extend the scale to 75°.

From 25° to 40° the scale follows the curve of natural sines; be-

FIG. 523.



yond 40° the increased intensity of shade is produced by keeping the same space between the hachures as for 40°, while increasing the space in which a certain number of hachures are placed by 25 per cent for a 45° slope, 50 per cent for a 55° slope, 75 per cent for a 65° slope, and 100 per cent for a 75° slope. For slopes less than 5° the breadth of hachures is the same as for 5°, but the 5° space is increased 25 per cent for 4°, 50 per cent for 3°, 75 per cent for 2°, and 100 per cent for 1°.

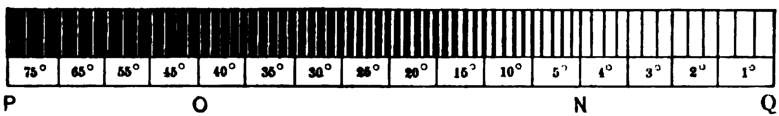
SLOPE.	RATIO OF BLACK TO WHITE.
1°	1 : 21
2	1 : 18
3	1 : 15⅓
4	1 : 12¾
5	1 : 8
10	2 : 7
15	3 : 6
20	4 : 5
25	5 : 4
30	3 : 2
35	7 : 3
40	4 : 1
45	5¼ : 1
55	6½ : 1
65	7¾ : 1
75	9 : 1

The ratio of black to white, according to this scale, is given in the table.

The number of hachures per inch for maps is 40 for a scale $\frac{1}{10000}$; 50 for $\frac{1}{20000}$; 65 for $\frac{1}{40000}$; 80 for $\frac{1}{60000}$.

To construct a working scale 20 hachures to the inch, divide N O (Fig. 524), 2 inches long, into 8 equal parts, erect perpendiculars, and draw two parallels to N O, one for spaces in which to number the slopes, the other of the depth desired for the length of

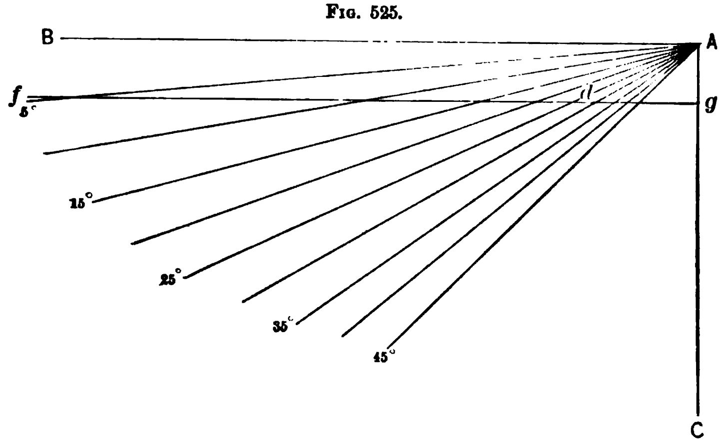
Fig. 524.



the hachures. Number these divisions from 5° to 40° and subdivide the upper rectangles into 5 equal parts by light vertical lines. Blacken each section in accordance with the scale, placing the width of the hachure always in the same direction from the vertical lines. From O toward the left and from N toward the right the spaces for the five hachures are to be increased in regular order by 25, 50, 75, and 100 per cent. A scale giving more hachures for each degree of slope should be constructed for a large map.

A scale for measuring the inclination of the slopes, as indicated by the contours, must be constructed before the hachures can be

Fig. 525.

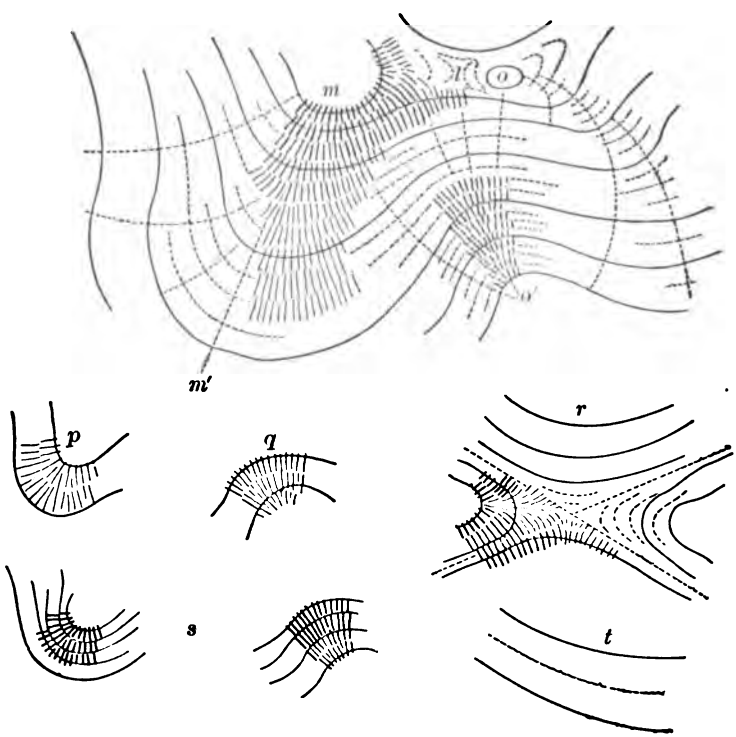


put in. Draw A B and A C (Fig. 525) perpendicular to each other. Draw through A lines making angles of 5°, 10°, 15° to 45° with A B; lay off A g equal to the contour interval to the scale of the map; draw $g f$ parallel to A B; cut off the paper along the line $g f$; by placing the point g in coincidence with any contour, the edge $g f$

being normal to it, the slope to an adjacent contour can be read from the scale determined by the intersection of the radial lines with gf. Instead of cutting the scale along gf, a pair of dividers may be used to transfer distances.

Putting in the Hachures.—They must be drawn very truly perpendicular to the contour lines. But if the contour lines are not parallel, the hachures must curve. In finishing drawings, sketch in

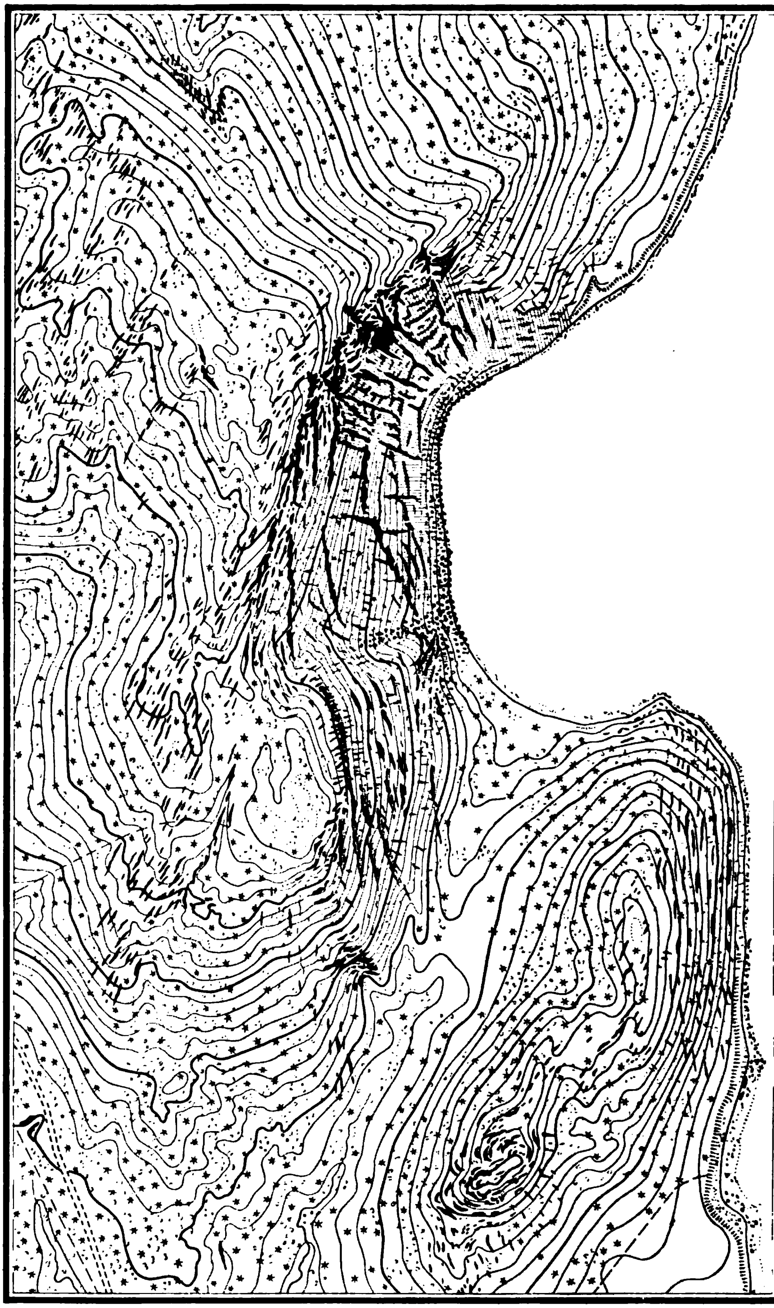
FIG. 526.



the curved hachures with a pencil at some distance apart as guides. When the contours are very far apart, as on nearly level ground, pencil in intermediate ones.

Hachures in adjoining rows should not be continuous, but "break joints," to indicate the places of the contour lines, which are usually penciled in to guide the hachures, and then rubbed out. The rows of hachures must neither overlap nor separate, and the lines should be made slightly tremulous. When they are put in

Fig. 527.



A Topographical Drawing of Eagle Cliff, by E. Hergesheimer, Assistant, United States Coast and Geodetic Survey. Scale $\frac{1}{10000}$.

without contour lines to guide them, take care never to let two rows run into one; for the breaks between the rows represent contour lines, and two contour lines of different heights can never meet, except on a vertical surface.

In drawing a hill begin at the top. When hachures diverge very much, as on hilltops, put in alternate short ones. When the formation is very convex or concave, short auxiliary contours may be used.

These points are illustrated in Fig. 526.

799. Brush Hill Shading. This method is used more as an adjunct of contours, although the French have worked out a scale of shades corresponding to different slopes, by means of which the surface can be represented to a scale, as with the horizontal and vertical methods.

As ordinarily employed, the steep surfaces are washed in with India ink to desired shades, corresponding to the assumption of either a vertical or oblique illumination.

Comparison of Systems.—Oblique illumination brings out rugged topography better than does the vertical illumination, while the latter produces better

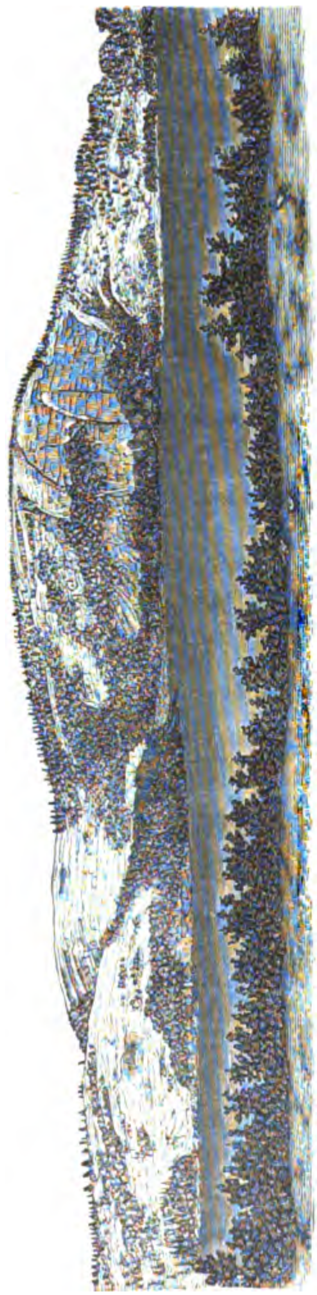


FIG. 528.

A View of Eagle Cliff, Mt. Desert Island (looking West).

effects on undulating surfaces. The vertical system of hachures seems to be preferred over the horizontal system, as being easier to draw and less liable to confusion with the details of the map; while for bringing out the configuration of a surface in a rapid manner the brush is preferable, though not giving the finish of pen work.

For fine plates showing the various systems of hill shading, see "Topographical Drawing," by I. Entoffer.

800. Topographical Mapping. Topographical mapping is the delineation or representation of the features of an area by conventional signs.

Drawing Paper.—Whatman's paper is usually used. There are three kinds: "Hot pressed" (H), which has a smooth surface, used for fine-line drawings; the "not hot pressed" (N), which has a finely grained surface, and is used for map work; and the rough (R), which has a coarsely grained surface suitable for strong-lined work and drawings on a large scale. The last two, which are called "cold pressed," are suitable for brush work. The following table gives names of different sizes of Whatman paper, and kinds of each size usually obtainable:

BRAND.	SIZE.	KINDS.			BRAND.	SIZE.	KINDS.		
Emperor.......	68" × 48"		N	R	Elephant...	28" × 33"	H	N	
Antiquarian.....	53 × 31	H	N		Superroyal..	27½ × 19¼	H	N	
Double Elephant.	40 × 26¾	H	N	R	Royal......	24 × 19¼	H	N	R
Atlas..........	34 × 26	H	N		Medium....	22 × 17	H	N	
Columbier......	34½ × 23½	H	N		Demy......	20 × 15	H	N	
Imperial.......	30 × 21				Cap........	17 × 13	H	N	

There are also fine expensive grades of imperial and double elephant known as extra weight. The side of the sheet nearest the eye, when the watermark reads aright, is that on which the drawing is usually made.

The Scale.—The choice of scale will depend upon (1) the size of the smallest details it is desired to represent upon the map, and (2) the advantage of having such a unit as will permit the employment of an ordinary triangular engineer's scale both in the construction and use of the map, and permit the conversion of actual and plotted distances with ease.

$\frac{1}{100}$ inch is as small a division as can be plotted or scaled from the plot; therefore a scale of 100 feet to an inch, or $\frac{1}{1200}$, is as small as can be used, and permit taking from the map distances to a single foot.

The English units not being based on a decimal system leads to two kinds of scales, those based on the inch and those on the foot.

For park work, scales 50 feet and 100 feet to the inch are used. For railroad work, scales 200 feet and 400 feet to the inch are used. For topographical surveys covering large areas, scales used are $\frac{1}{2500}$ to $\frac{1}{10000}$; $\frac{1}{10000}$ to $\frac{1}{25000}$.

The Location of the Map on the Paper.—To determine the size of paper needed for a given map, and its location upon the paper, add to the distance between the extreme east and west station points, as given by their co-ordinates, the distance that the topography has been taken beyond these points, as shown by the notes. Locate the middle of this distance in the center of the paper in an east and west direction, unless allowance must be made for title, meridian line, etc. Locate the center in a north and south direction in the same manner. From the co-ordinates of the triangulation station, that was used as the origin of co-ordinates for the triangulation, and from the co-ordinates of the western and southern limits of the map, this triangulation station may be located upon the paper. Before beginning to plot, two scales should be laid off upon the paper, one parallel to each axis of co-ordinates, in order to avoid the introduction of errors due to the expansion and contraction of the paper, produced by the varying humidity of the atmosphere.

If the map is to be made on several different sheets, these sheets should be numbered, and an index map be made, with the parts covered by the separate sheets numbered to correspond.

Plotting the Notes.—Two methods are used for plotting the station points:

1. By rectangular co-ordinates;
2. By polar co-ordinates.

The first method is used when the greatest accuracy is required; the second is employed in locating points of lesser importance.

1. *By Rectangular Co-ordinates.*—Since the triangulation is the base and control of the map, the triangulation stations should be plotted by the method of rectangular co-ordinates. The co-ordinates for each station having been determined in the computation of the triangulation, they are used for this purpose. The sheet should be divided into a network of squares whose sides represent 100 to 200 units. The scale is then used within the limits of these squares. To construct the squares: Draw a line at the lower edge of the paper; erect perpendiculars to it near either end; lay off the same distance on each perpendicular; join the tops and scale the distance between these points, if it is the same as between their bases; subdivide the opposite sides and draw lines through the points of division. In locating a point in a square draw lines across it from measured distances on its sides. After plotting the triangulation stations by co-ordinates, check by scaling the distances between the stations, to see that they agree with the computed lengths.

2. *By Polar Co-ordinates.*—In plotting by polar co-ordinates the directions of the lines may be determined either by the use of a protractor or by natural tangents.

Steel protractors are made, with verniers for setting off angles to minutes, and a scale of equal parts upon the arm for plotting the distances. The Colby protractor, made by Keuffle & Esser, has the advantage that the rotating part does not rest upon the paper, thus avoiding soiling it; and it is held in position by weights, thus avoiding puncturing the paper. Very rapid work can be done with its use, 216 shots per hour during a period of twenty-five hours and a half having been plotted without difficulty, with one man to call off while the other plotted.*

Paper protractors are printed upon drawing paper and Bristol board. The most accurate are those printed directly upon the drawing paper that is to be used for mapping. Directions are transferred by means of triangles or parallel ruler. If the map is large, or it is undesirable to have the printed protractor upon it, cut out the center of the protractor sheet and place it over the station point upon the map by means of two lines drawn through

* Journal of Association of Engineering Societies, September, 1896, vol. xvii, No. 3.

the point, one parallel to the east and west line, and the other parallel to the north and south line.

Plotting with a protractor has the advantage of rapidity. It has the disadvantage that an error in plotting one station affects all subsequent stations. All courses should be plotted by their azimuth, and not by their deflection from the preceding course; this avoids a cumulative error in azimuth.

Another disadvantage of plotting by a protractor lies in not being able to tell how much of the error of closure is due to plotting, and how much to field work.

Protractors should be held in position by weights, and not pinned down through the map.

Azimuths may be accurately plotted by means of natural tangents. Select a radius of ten to save multiplication of tangents given, and of such a length that the point to be plotted will fall inside the tangent, so that any error in laying off the tangent will be divided where the station is to be plotted instead of being multiplied, as it would be if the radius had to be prolonged to the station point.

By using the triangular boxwood rule six different lengths of radii may be had.

To locate errors either in plotting or field work, if the meander does not close, plot backward to the starting point. Two coincident lines would indicate an error in the length of that line, either in field work, reduction of notes, or plotting.

If the error can not be located by plotting, compute the co-ordinates of the points, noting the error in closure and determining its azimuth. The error in longitude divided by the error in latitude will be the tangent of the azimuth of the line that will close the meander. If this azimuth is nearly equal to that of any one of the courses the error may be found in that one.

If in plotting topography points, the elevation of the point located is so placed upon the map as to make the point serve as the decimal point in its elevation, much confusion and multiplication of lines will be avoided.

Plot no topography from a stadia point until the meander of which that stadia point is a part is closed and adjusted.

Begin with points determining roads, streams, bodies of water, etc.; then locate buildings and boundaries, marking out the area having different kinds of vegetation, putting in last points for the location of contours, and then the contours themselves.

It is a good plan to erase all construction lines, meridian lines for locating protractor, etc., before beginning to ink. In inking the map follow the same order observed in penciling it.

Completing the Map.—To complete the map after all the topographical features are represented, it requires:

1. The *Lettering*—names of places, streets, title, etc. The United States Coast and Geodetic Survey employ Roman letters only in lettering maps, using the Italics for all water features.

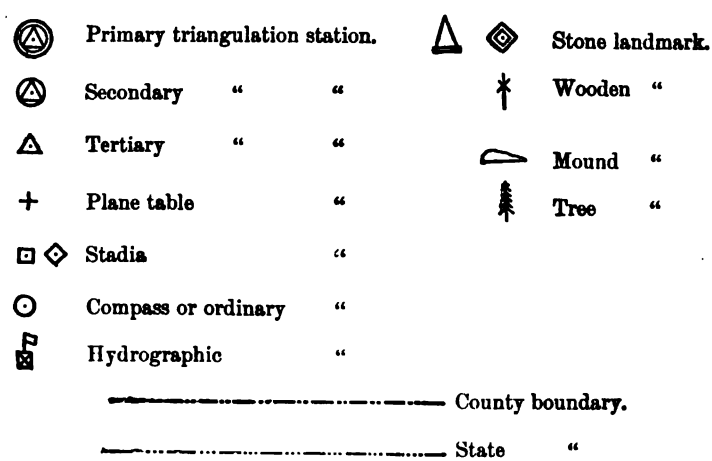
According to the importance of points marked, either all capitals, small letters with initial capitals, or small letters are used.

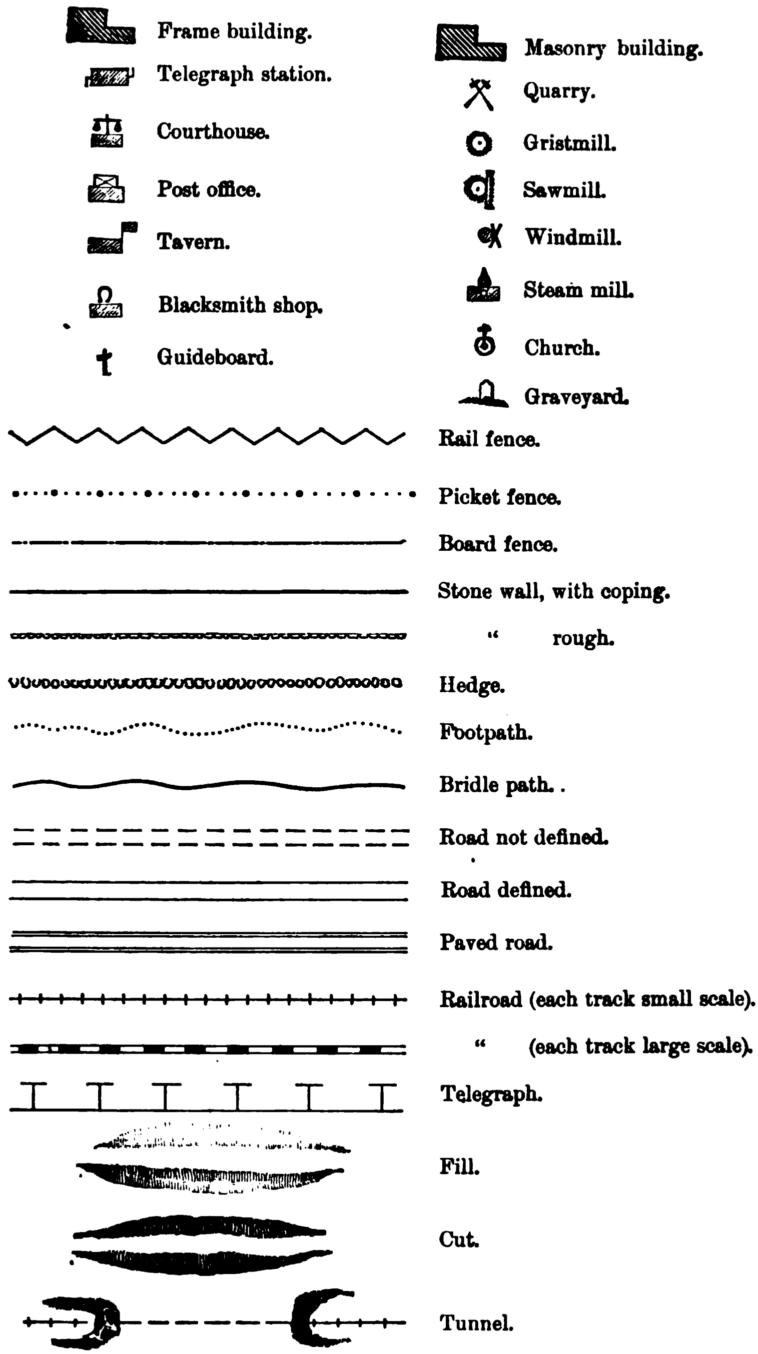
2 A *Scale.*—In addition to recording the ratio of the mapped distances to the actual ones, a short section of the scale employed should be constructed and properly marked for convenience in taking off distances from the map.

3. On small maps it is necessary to indicate the direction of the true north, and the magnetic variation as well; on large maps having meridians and parallels of latitude a record of the variation is sufficient.

801. Conventional Signs.

Miscellaneous Signs.

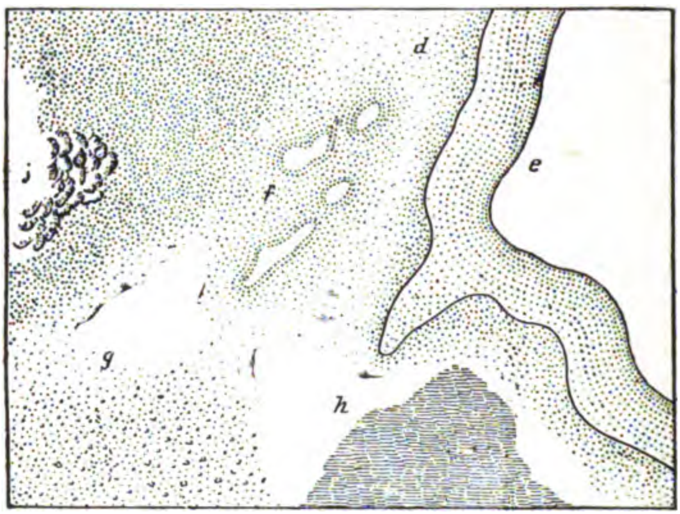



Frame building.
Masonry building.
Telegraph station.
Quarry.
Courthouse.
Gristmill.
Post office.
Sawmill.
Tavern.
Windmill.
Steam mill.
Blacksmith shop.
Church.
Guideboard.
Graveyard.
Rail fence.
Picket fence.
Board fence.
Stone wall, with coping.
" rough.
Hedge.
Footpath.
Bridle path.
Road not defined.
Road defined.
Paved road.
Railroad (each track small scale).
" (each track large scale).
Telegraph.
Fill.
Cut.
Tunnel.

Sand.—Sand is represented by fine dots evenly distributed over the surface. Along shores the dots are arranged in lines parallel to the water line between high and low water; a full line marks high-water line. Sand dunes are represented at *f*, in Fig. 529.

Gravel is represented by coarse dots and small curved and angular outlines (shown at *g*).

FIG. 529.



Mud is represented by short strokes in sets, their direction being parallel to the bottom of the map (shown at *h*).

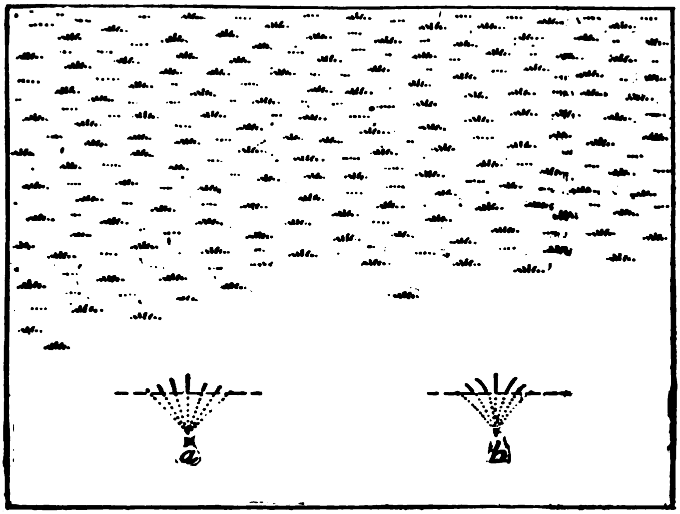
Rocks are drawn in their proper places imitating their true appearance, as seen from above, by irregular angular forms (shown at *j*).

Grass (Fig. 530) is represented by groups of from five to seven fine dots and dashes representing tufts of grass distributed evenly over the surface. The base of each tuft should be parallel to the lower edge of the map, while the dashes increasing in length from the ends to the center will give each tuft a rounded top. The dashes should be drawn in a direction radiating from a point below the middle of the base.

The signs should be distributed over the surface so as to produce a flat tint at a distance; to obtain this effect the signs must not be placed in rows. Groups of dots in light spots will correct

an uneven distribution of tufts, and if the T square be used as a support for the hand while putting in the signs, its upper edge will serve as a guide to make the base of the tufts horizontal.

FIG. 530.



To distinguish between grass land that has been under cultivation—that is, meadow land—and cleared land, the signs in the former are made in regular rows.

FIG. 531.

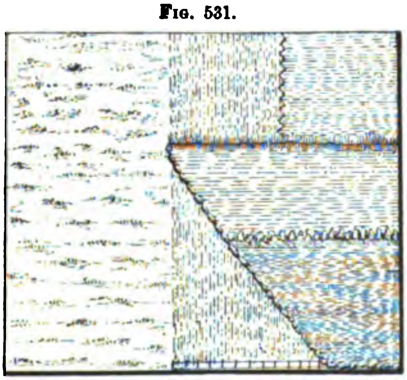


FIG. 532.

Cultivated land (Fig. 531) is shown by alternate broken and dotted lines representing the furrows, or rows of short dashes at

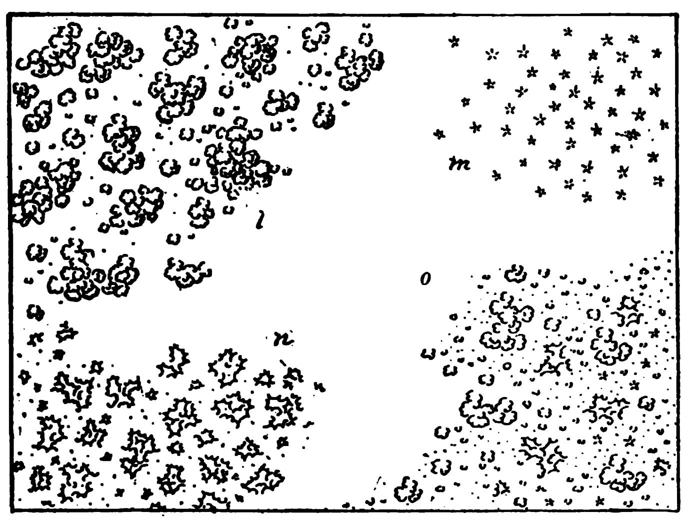
right angles to the line, the form indicating the rows of field products. Dashes should be short, not longer than the distance required for three dots in a dotted line; the space between dashes in the same row should be of the same length as the dash, and the dash in one row opposite the space in the adjacent row.

To give greater prominence to the division into fields, in adjacent fields the lines are drawn in different directions, usually parallel to one of the sides.

The United States Coast and Geodetic Survey have used special signs for the representation of special field products on maps upon a large scale. These signs are suggestive of the product.

Gardens (Fig. 532) are divided into areas separated by spaces representing paths. These areas are filled with signs representing

FIG. 533.



cultivation, their lines running in different directions in adjoining areas.

Trees (Fig. 533), in general, are represented by a cluster of scalloped curves, convex outward, giving the appearance of the tree in plan. To give relief, the curves on the lower right-hand side are made heavier, in conformity with the usual assumption of light coming from the upper left hand at an angle of 45° (at *l*).

If but a few trees are scattered over a surface they are sometimes drawn "in elevation" (Fig. 534).

If different kinds of trees are to be distinguished, the conventional sign for trees in general is used to designate trees with de-

Fig. 534.

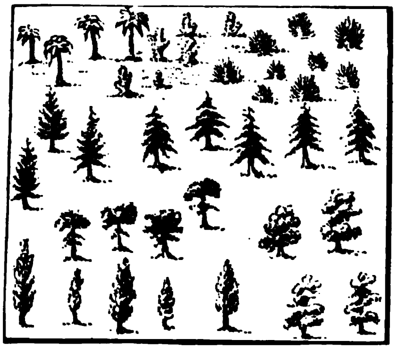
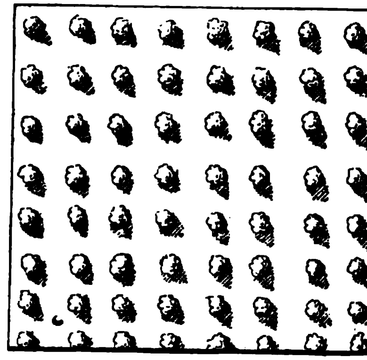


Fig. 535.



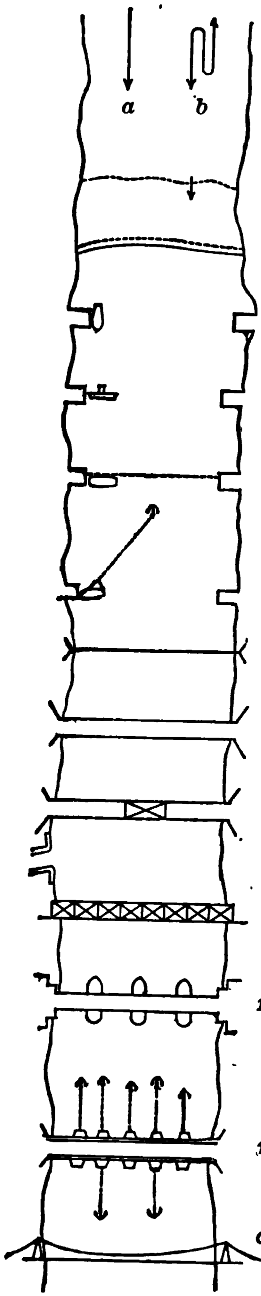
ciduous foliage, excepting the oak, for which the scalloped curves are concave outward (at n). Pines and trees with evergreen foliage are designated by starlike forms (Fig. 553 at m).

Mangrove and palmetto have been given special forms, the former representing the interlacing of the roots visible above the surface, the latter by a whorl of long leaves.

Woods are represented by a collection of signs of the various kinds of trees composing them. The best effect is obtained by varying the size of the trees, shading them, but not representing their shadows. If underbrush or thicket is to be represented, small curves, incomplete and small signs for trees, and dots, are used (at o).

Orchards (Fig. 535) are represented by trees in plan, as described above for trees in general; but the trees are arranged in regular rows and their shadows constructed, the light coming from the upper left hand. The shadow is constructed of a series of parallel lines drawn perpendicular to the direction of the light, the outline of the shadow being of an oval or pointed form, care being taken not to have it extend beyond the space inclosed by lines tangent to the outline of the tree and parallel to the light.

Shadows are not constructed for trees except as they are set out, as in the case of orchards, or trees set out for foliage.



a Direction of flow of current.

b Current alternating in direction.

c Ford, foot or horses only.

d " carriages.

e Ferry, rowboat.

f " steam.

g " rope.

h " flying.

i Bridge, foot.

j " wooden girder.

k " draw.

l " truss.

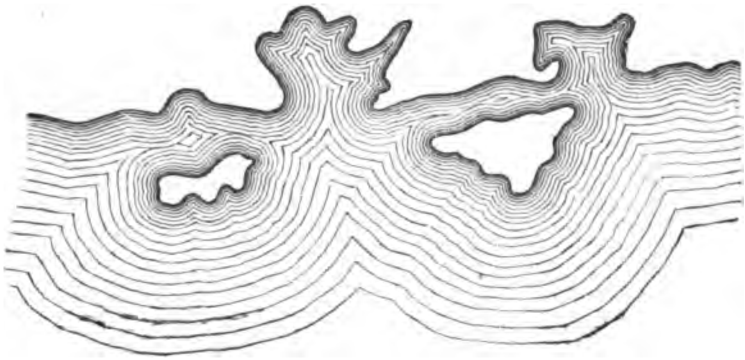
m " masonry.

n " pontoon.

o " suspension.

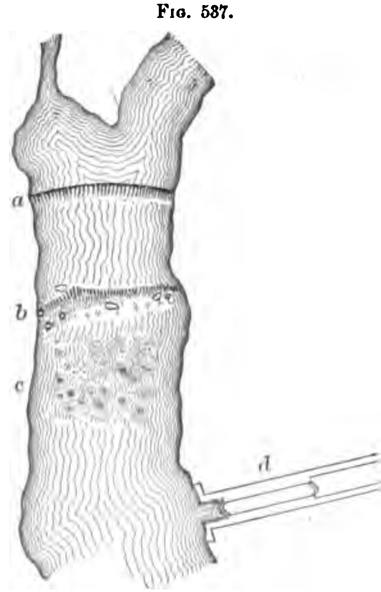
Clumps of shrubs and bushes set out are represented as a mass of foliage of irregular outline, instead of having the regular form that trees have.

Fig. 536.



Water is represented in different ways. Along a *lake* or *seacoast* water is represented by a series of parallel lines conforming to the windings of the shore; the distance between the lines increasing and the width of the line decreasing as they are farther from the shore (Fig. 536).

Fig. 537.



Rivers have lines drawn parallel to each shore, increasing in their spacing toward the center, the entire distance between the banks being filled with lines, if both banks are shown on the map (Fig. 537). If but one bank is shown, it is represented as described for a lake or seacoast.

Brooks are represented by a single line of increasing width, or by two lines very gradually divergent, depending upon the size of the brook.

A *dam* is shown at *a*; a *falls* at *b*; a *canal* and *lock* at *d*; and *rapids* at *c* (Fig. 537).

Ponds have their surface covered by fine lines parallel to the

lower edge of the map (Fig. 538). Sometimes they have their surface filled by lines parallel to the shore, as for lakes (Fig. 539).

Marshes are represented by a combination of tufts of grass, and horizontal lines for water. In a fresh-water marsh the lines are short and irregular (Fig. 539). In a salt marsh they are continuous

Fig. 538.

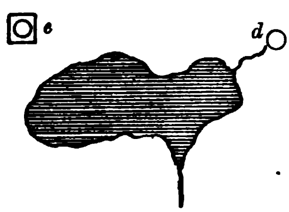
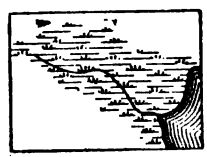
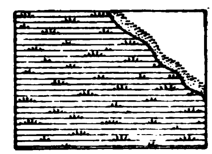


Fig. 539.



Fresh marsh and pond.

Fig. 540.



Salt marsh.

(Fig. 540). The deeper the water in the marsh the shorter the grass tufts and the more prominent the lines.

The United States Geological Survey put in all the above lines, denoting water with strong Prussian blue.

A *spring* is shown at *d*, and a *well* at *e* (Fig. 538).

802. Colored Topography. Water colors are sometimes used to fill in the topography of a map. Their use has the advantage of great rapidity and the possibility of strongly contrasting parts. The lack of permanence of some colors, notably Prussian blue and crimson lake, makes necessary the tinting of a small space properly marked, as a key to the map.

The conventional tints are as follows*:

Sand.—Yellow ocher, a flat tint. Gravel may be shown by dots of burnt sienna.

Water.—Prussian blue, a flat tint. If a lower tone is desired for the map, use indigo instead of Prussian blue. The surface is also outlined with a strong blue.

Cultivated Land.—Burnt sienna, a flat tint, which when dry is ruled with a stronger tint of the same color. To give variety, alternate fields may be tinted with Payne's gray and crimson lake or burnt sienna and crimson lake.

* "Topographical Drawing and Sketching," Lieut. H. A. Reed, U. S. A.

Cleared Land.—Indigo and gamboge, a warm flat tint.

Underbrush.—The tints for cleared and cultivated land alternating in irregularly shaped patches, and blending into each other.

Marsh.—Prussian blue "dragged" (i. e., applied with the side of the brush to give a rugged appearance), the strokes of the brush being parallel to the lower edge of the map. The lower edges of the strips of land are then shaded with a strong tint of Payne's gray and crimson lake, applied with the tip of the brush and parallel to the bottom of the map.

Mud.—The same horizontal stroke as in pen drawing, except that sepia is used instead of India ink.

Trees.—Trees and clumps of trees are first outlined in pencil; a flat tint of gamboge laid over the surface within these outlines; the shading is then effected with a green tint stronger and cooler than that used for cleared land, composed of indigo and gamboge or Prussian blue and gamboge, according as indigo or Prussian blue is used for water. The rounded form of the tree is brought out by shading the side opposite the source of light with curved strokes, as with the pen.

Shadows are represented by a tint of Payne's gray and crimson lake.

For maps to a small scale, a flat tint of Prussian blue and gamboge, with a little sepia, is recommended for the entire surface within the penciled outlines of the mass of trees. This sign is always superposed on that of the general surface.

Buildings of wood are outlined and shaded with India ink, then tinted with sepia; of masonry, a strong tint of crimson lake is used for outlining and shading, while a lighter tint is washed over the surface. Payne's gray may be used for stone, to distinguish from brick buildings, or the walls may be made heavier, as in pen work.

Roads.—Penciled outlines are filled with a flat tint of yellow ocher, the edges being afterward traced in India ink.

Bridges.—The outline is filled with yellow ocher, and wood and masonry are distinguished as in buildings.

Fences are represented as in pen drawing. *Stone walls* have their outline filled with crimson lake.

Slopes.—To emphasize slopes, graded tints of Payne's gray and India ink or cobalt on ivory black are washed in parallel to the contours.

Contours are drawn with strong crimson lake, the reference being given in India ink or red.

Rocks.—Sepia warmed with a little burnt sienna is dragged over the surface, giving the general outline; by going over parts of the surface two or three times the required effect may be obtained.

CHAPTER XVII.

MARITIME OR HYDROGRAPHIC SURVEYING.

INTRODUCTION.

803. THE object of hydrographic surveying is to fix the positions of the deep and shallow points in harbors, rivers, etc., and thus to discover and record the contour of the surface under the water, the shoals, rocks, channels, and other important features of the locality.

The surveys on the water are usually based upon previously determined points on the shore. In surveys of any importance, the relative positions of prominent points on the shore are first very precisely determined by Trigonometric Surveying, Chapter X. These form the basis of operations, and afford the means of correcting the results obtained by the less accurate methods employed for filling in the details.

The nature of the work makes special instruments and methods necessary.

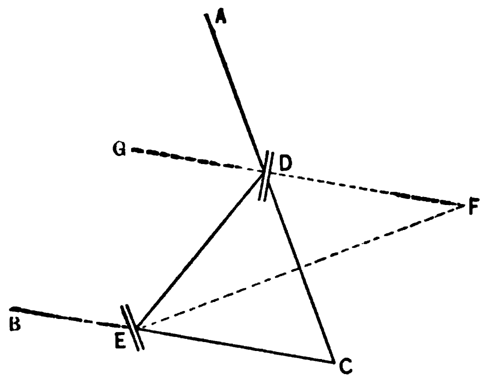
In addition to the surveying instruments already described, the sextant is much used in hydrographic surveying. When the sextant is used for determining the position of a point, the angles are measured between three lines, passing from the required point to three known points. The required point is thus determined by trilinear co-ordinates, or by the *fifth method*, as explained in Article 8, Part I.

The Sextant.

804. Principle. The angle subtended at the eye by lines passing from it to two distant objects may be measured by so arranging two mirrors that one object is looked at directly, and the other object is seen by its image reflected from one mirror to the second, and from the second mirror to the eye. If the first mirror be

moved so that the doubly reflected image of the second object be made to cover or coincide with the object seen directly, then is the desired angle equal to twice the angle which the mirrors make with each other.

Fig. 541.



Proof.—In Fig. 541, let D and E be two mirrors perpendicular to the plane of the paper. Let a ray of light from the object A be reflected from the mirrors D and E to the eye at C, and B be the other object, looked at directly. Erect perpendiculars to the mirrors, and prolong them until they meet at F. Prolong the line A D until it meets the line B E at C. The angle D F E is equal to the angle which the two mirrors make with each other.

Since the angle of incidence equals the angle of reflection,

$$ADG = GDE, \text{ and } DEF = FEC.$$

then we have:

$$DCE = ADE - DEC$$
$$DCE = 2\,(GDE - DEF)$$
$$DCE = 2\,DFE$$

805. Description of the Sextant (Fig. 542). The frame is usually of brass, constructed so as to combine strength with lightness. The handle by which it is held is of wood. The index arm is movable about a pivot in the center of the graduated arc. The index glass is a small mirror, attached to the index arm at the pivot, so as to be perpendicular to the plane of the graduated arc. The horizon glass on the left in the figure is attached perpendicularly to the plane of the instrument, and parallel to the index glass when the index is at 0. The lower half of this glass is silvered, to make it a reflector, and the upper half is transparent. The telescope is attached so as to point toward the horizon glass. Sets of colored

glasses are used to moderate the light of the sun, when that body is observed.

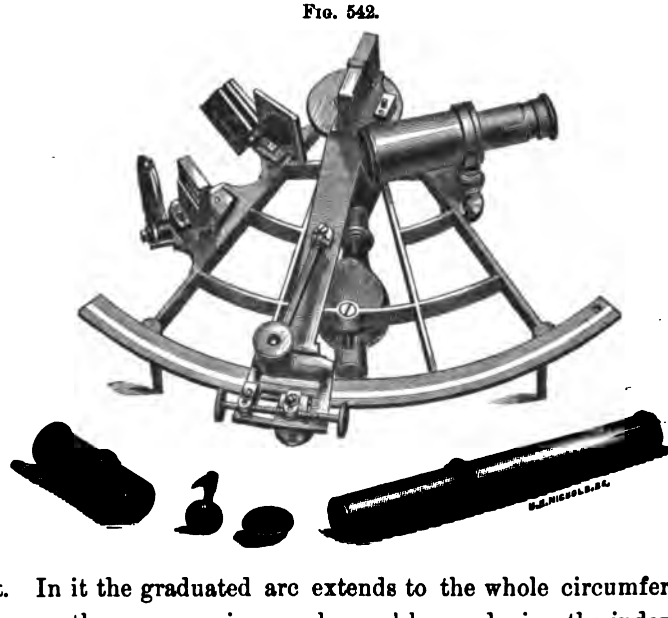
The sextant has an arc of one sixth of a circle, and measures angles up to 120°, the divisions of the graduated arc being numbered with twice their real value, so that the true desired angle, subtended by the two objects, is read off at once. The arc is usually graduated to 10′, and read by a vernier to 10″.

806. The box or pocket sextant has the same glasses as the larger sextant, inclosed in a circular box about three inches in diameter. The lower part, which answers for a handle when in use, screws off and is used for a cover.

The octant has an arc of one eighth of a circumference, and measures angles to 90°.

807. The Reflecting Circle. This is an instrument constructed on the same principle and used for the same purposes as the sex-

FIG. 542.



tant. In it the graduated arc extends to the whole circumference, and more than one vernier may be used by producing the index arm to meet the circumference in one or two more points.

808. Adjustments of the Sextant. 1. *To make the index glass perpendicular to the plane of the arc:* Bring the index near the center of the arc, and place the eye near the index glass and nearly in the plane of the arc. See if the part of the arc reflected in the mirror appears to be a continuation in the same plane of the part seen directly. If so, the glass is perpendicular to the plane of the arc. If not, adjust it by the screws behind it.

2. *To make the horizon glass perpendicular to the plane of the arc:* Hold the instrument vertically, and bring the direct and reflected images of a smooth portion of the distant horizon into coincidence; then turn the instrument until it makes an angle with the vertical. If the two images still coincide, the glasses are parallel; and, as the index glass has been made perpendicular to the plane of the arc, the horizon glass is in adjustment. If the images do not coincide, the horizon glass must be adjusted by the adjusting screw.

3. *To make the line of collimation of the telescope parallel to the plane of the arc:* The line of collimation of the telescope is an imaginary line, passing through the optical center of the object lens and a point midway between the two parallel wires. These wires are made parallel to the plane of the sextant by revolving the tube in which they are placed.

To see whether the line of collimation of the telescope is in adjustment, bring the images of two objects, such as the sun and moon, into contact at the wire nearest the instrument, and then, by moving the instrument, bring them to the other wire. If the contact remains perfect, the line of collimation is parallel to the plane of the arc; if it does not, the adjustment must be made by the screws in the collar of the telescope.

4. *To see if the two mirrors are parallel when the index is at zero:* Bring the direct and reflected images of a star into coincidence. If the index is at zero, then no correction is necessary; if not, the reading is the "*index error*," and is positive or negative, according as the index is to the right or left of zero.

The index error may be rectified by moving the horizon glass until the images do coincide when the index is at zero, but it is usually merely noted, and used as a correction, being added to

each reading if the error is positive, or subtracted from each reading if the error is negative.

809. How to observe. Hold the instrument so that its plane is in the plane of the two objects to be observed, and hold it loosely. Look through the eyehole, or plain tube, or telescope, at the left-hand or lower object, by direct vision, through the unsilvered part of the horizon glass. Then move the index arm till the other object is seen in the silvered part of the horizon glass and the two are brought to apparently coincide. Then the reading of the vernier is the angle desired.

If one object be brighter than the other, look at the former by reflection. If the brighter object be to the left or below, hold the instrument upside down.

If the angular distance of the object be more than the range of the sextant (about 120°), observe from one of them to some intermediate object, and thence to the other.

A good rest for a sextant is an ordinary telescope clamp, through which is passed a stick, one end of which is fitted into a hole made in the sextant handle, and the other end of which is weighted for a counterpoise.

THE PRACTICE.

810. To set out Perpendiculars. Set the index at 90°. Hold the instrument over the given point by a plumb line, and look along the line by direct vision. Send a rod in about the desired direction, and when it is seen by reflection to coincide with the point on the line looked at directly, it will be in a line perpendicular to the given line at the desired point.

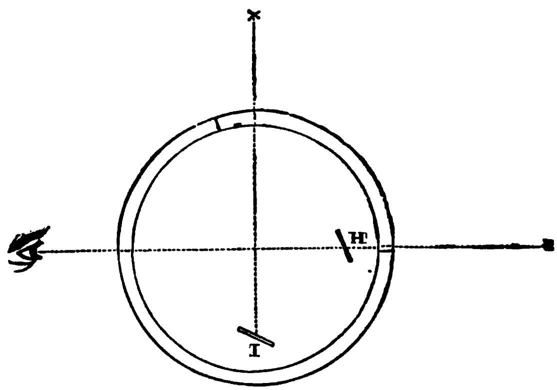
Conversely, to find where a perpendicular from a given point would strike a line:

Set the index at 90°, and walk along the line, looking directly at a point on it, until the given point is seen by reflection to coincide with the point on the line. A plumb line let fall from the eye will give the desired point.

811. The Optical Square (Fig. 543). This is a box containing two mirrors, fixed at an angle of 45° to each other, and therefore

giving an angle of 90°, as does the sextant with its glasses fixed at that angle. It is used to set out perpendiculars.

FIG. 543.

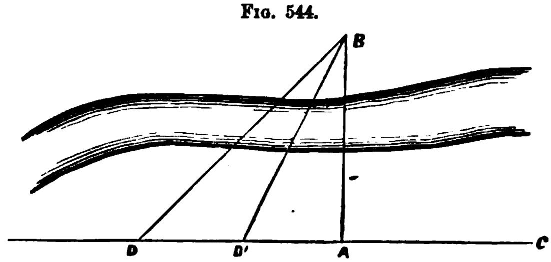


812. To measure a Line, One End being Inaccessible (Fig. 544). Let A B be the required line, and B the inaccessible point.

At A set off a perpendicular, A C, by Art. 810; then set the index at 45°, and walk backward from A in the line of C A prolonged, looking by direct vision at C, until you arrive at some point, D, from which B is seen by reflection to coincide with C. Then is A D = A B.

If more convenient, after setting off the right angle, set the index at 63° 26′, and then proceed as before. The objects will be

FIG. 544.



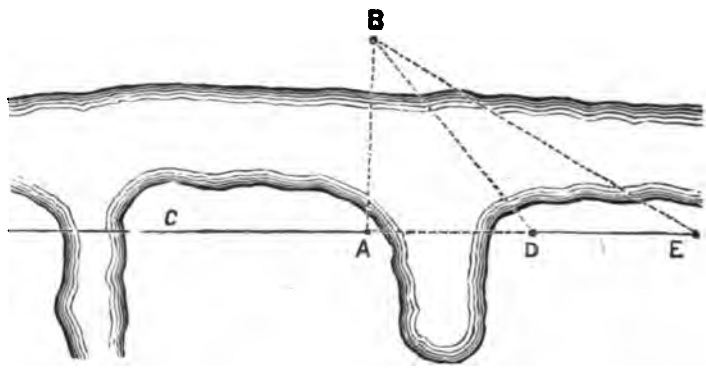
seen to coincide when at some point, D′. Then A D′ = $\frac{1}{2}$ A B. If the index be set at 71° 34′, then the measured distance will be $\frac{1}{3}$ A B, and so on.

If the index be set at the complements of the above angles, the

distance measured will be, in the first case, twice, and in the second case three times the desired one.

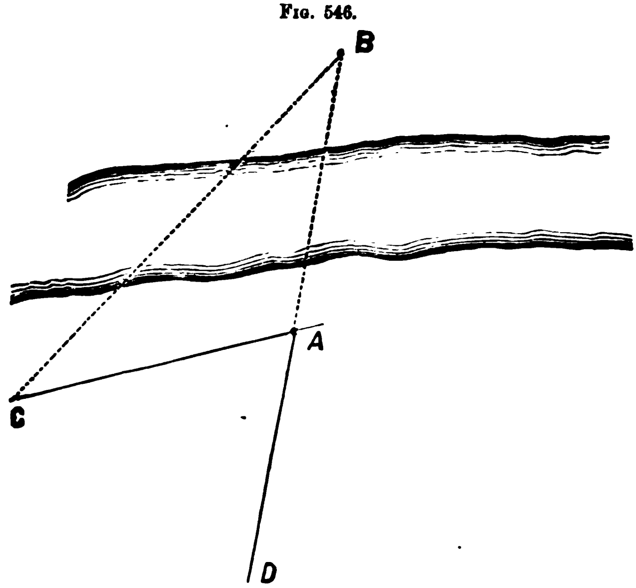
When the distance A D can not be measured, as in Fig. 545, fix

FIG. 545.



D as before. Set the index at 26° 34′, and go along the line to E, where the objects are seen to coincide with each other; then is A E twice A B, and hence E D = A B.

FIG. 546.

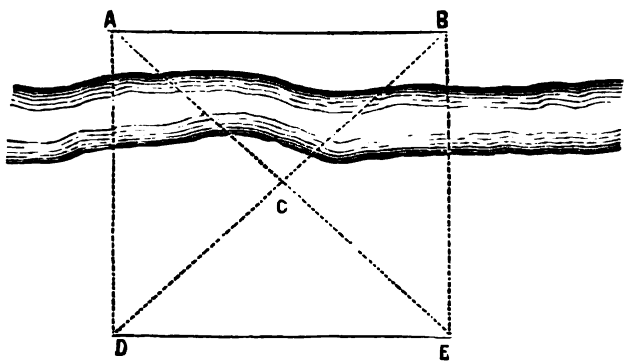


Otherwise (Fig. 546). At A set off an angle, as C A D (A D being a prolongation of A B); then walk along the line A C with

the index set to half that angle, looking at A directly, and B by reflection, till you come to some point, C, at which they coincide. Then is C A = A B.

813. To measure a Line when Both Ends are Inaccessible (Fig. 547). Let A B be the required line. At any point, C, measure the angle A C B. Set the sextant to half that angle, and walk back in

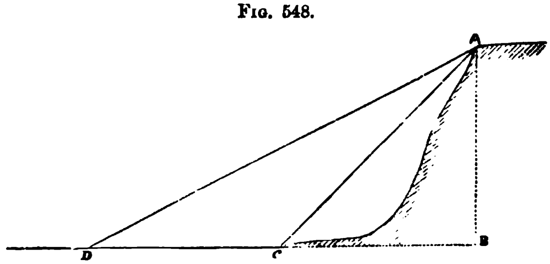
Fig. 547.



the line B C prolonged till at some point, D, A and B are seen to coincide, as in the last problem; thus making A C = C D. Do the same on A C produced to some point, E. Then is D E = A B.

All the methods for overcoming obstacles to measurement, determining inaccessible distances, etc. (Part I, Chapter V), with the transit or theodolite, can be executed with the sextant.

Fig. 548.



814. To measure Heights. Measure the vertical angle between the top of the object and a mark at the height of the eye, as with a theodolite or transit, and then calculate the height by trigonometry.

Otherwise (Fig. 548). Set the index at 45°, and walk backward till the mark and the top of the object are brought to coincide. Then the horizontal *distance* equals the *height.*

So, too, if the index is set at 63° 26′, the height equals twice the distance, and so on. The ground is supposed to be level.

When the base is inaccessible: Make C = 45°, and D = 26° 34′. Then C D = A B. So, too, if C = 26° 34′, and D = 18° 26′.

This may be used when a river flows along the base of a hill whose height is desired, or in any other like circumstance.

815. To observe Altitudes in an Artificial Horizon. In this method we measure the angle subtended at the eye between the object and its image reflected from an artificial horizon. An artificial horizon may be a small mirror, provided with a level for adjusting, or, better still, a shallow tray of wood or metal for holding a fluid the surface of which will act as a reflector. Mercury, oil, molasses, water, or some other fluid may be used. The artificial horizon most used is a tray six inches long, three inches wide, and three quarters of an inch deep, holding mercury. When not in use the mercury is kept in an iron flask. To shield the surface of the mercury from wind, a cover with a sloping glass top is placed over the tray, as shown by the dotted lines in Fig. 549. The image of the object in the mercury is looked at directly, and the object itself is viewed by reflection. The object observed is supposed to be so distant that the rays from it, which strike respectively the index glass and the artificial horizon, are parallel—i. e., S and S′ (Fig. 549), are the same point.

Then will the observed angle S E S″ be double the required angle S E H.

Demonstration.

$$a = a',\ a' = a'',\ \text{and}\ a'' = a'''.\quad \text{Hence}\ a''' = a.$$
$$\text{S E S}'' = a + a''' = 2\,a = 2\ \text{S E H}$$

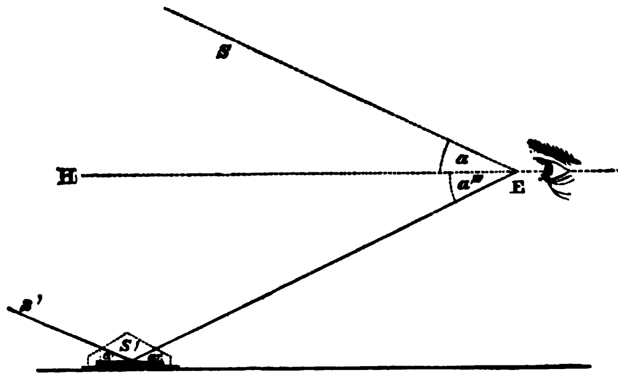
When the sun is the object observed, to determine whether it is his upper or lower limb whose altitude has been observed, proceed thus:

Having brought two limbs to touch, push the index arm from you. If one image passes over the other, so that the other two

limbs come together, then you had the lower limb at first. If they separate, you had the upper limb.

In the forenoon, with an inverting telescope, the lower limbs are

FIG. 549.

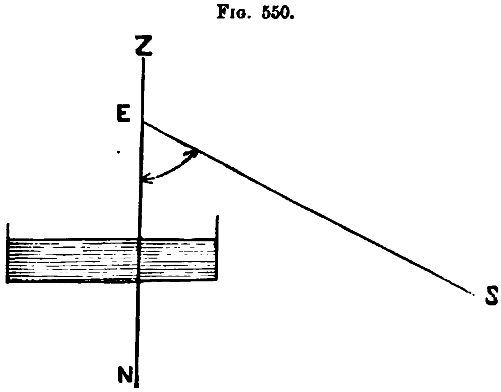


parting, and the upper limbs are approaching; and *vice versa* in the afternoon.

To observe very small altitudes and depressions with the artificial horizon:

Stretch a string over the artificial horizon. Place your head so that you see the string cover its image in the mercury. Then the eye and string determine a vertical plane.

FIG. 550.



Then observe, looking at the string by direct vision, and seeing the object by reflection, and you have the angle S E N, in Fig. 550, the supplement of the zenith distance.

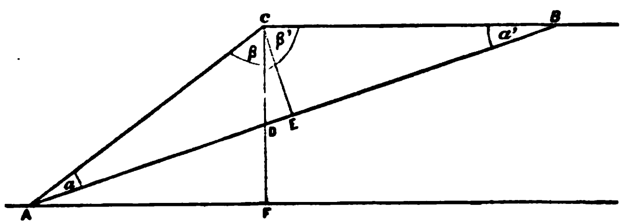
Otherwise. Fix behind the horizon glass a piece of white paper with a small hole in it, and with a black line on it perpendicular to the plane of the arc.

Then look into the mercury, so as to see in it the image of the line. Your line of sight is then vertical, and the angle to the object seen by reflection is measured as before.

316. To measure Slopes with the Sextant and Artificial Horizon. Let A B, Fig. 551, be the surface of the ground, and A F a horizontal line. Mark two points equally distant from the eye. Measure, by the preceding method, the angles β and β', which C A and C B make with the vertical C D. Then will half the difference of these angles equal the angle which the slope makes with the horizon.

Demonstration. Continue the vertical line C D to meet the horizontal line in F, and draw C E perpendicular to A B. Then

FIG. 551.



the triangles C D E and A D F are similar, being right-angled and having the acute angles at D equal. Consequently, the angle D C E = D A F, which is the angle of the slope with the horizon. But D C E $= \frac{1}{2}(\beta' - \beta)$, hence $\frac{1}{2}(\beta' - \beta) =$ the angle which the slope of the ground makes with the horizon.

If the points A and B be not equally distant from C, but yet far apart, this method will still give a very near approximation, the error, which is additive, being $\frac{1}{2}(a' - a)$.

Demonstration.

$$\begin{aligned}
\text{D C E} &= \beta' + a' - 90^\circ \\
\text{D C E} &= -\beta - a + 90^\circ \\
\hline
2\,\text{D C E} &= \beta' - \beta + a' - a \\
\text{D C E} &= \tfrac{1}{2}(\beta' - \beta) + \tfrac{1}{2}(a' - a)
\end{aligned}$$

Oblique Angles.—When the plane of two objects, observed by the sextant, is very oblique to the horizon, the observed angle will

differ much from the horizontal angle which is its horizontal projection, and which is the angle needed for plotting. The projected angle may be larger or smaller than the observed angle.

This difficulty may be obviated in various ways:

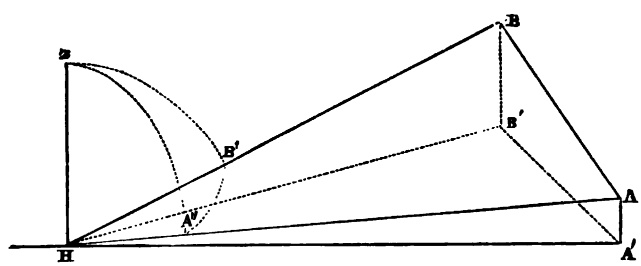
1. Observe the angular distance of each object from some third object, very far to the right or left of both. The difference of these angles will be nearly equal to the desired angle.

2. Note, if possible, some point above or below one of the objects, and on the same level with the other, and observe to it and the other object.

3. Suspend two plumb lines, and place the eye so that these lines cover the two objects. Then observe the horizontal angle between the plumb lines.

4. For perfect precision, observe the oblique angle itself, and also the angle of elevation or depression of each of the objects.

FIG. 552.



With these data the oblique angle can be reduced to its horizontal projection, either by descriptive geometry or more precisely by calculation, thus:

Let A H B, Fig. 552, be the observed angle, and A′ H B′ the required horizontal angle.

Conceive a vertical H Z, and a spherical surface, of which H, the vertex of the angle, is the center. Then will the vertical planes, A H A′ and B H B′, and the oblique plane, A H B, cut this sphere in arcs of great circles, Z A″, Z B″, and A″ B″, thus forming a spherical triangle, A″ Z B″, in which A″ B″ $= h$ measures the observed angle; Z A″ $=$ Z measures the zenith distance of the point A; and Z B″ $=$ Z′ measures the zenith distance of the point B.

These zenith distances are observed directly, or given by the

observed angles of elevation or depression. Then we have the three sides of the triangle to find the angle B = A′ H B′.

Calling P the half sum of the three sides, we have

$$\text{Sin.}\,\tfrac{1}{2}\,B = \sqrt{\frac{\sin.\,(P - Z)\sin.\,(P - Z')}{\sin.\,Z\,.\,\sin.\,Z'}}$$

An approximate correction, when the zenith distances do not differ from 90° by more than 2° or 3°, is this:

$$\left(90^\circ - \frac{Z + Z'}{2}\right)^2 \tan.\,\tfrac{1}{2}\,h\,.\,\sin.\,1'' - \left(\frac{Z - Z'}{2}\right)^2 \cot.\,\tfrac{1}{2}\,h\,.\,\sin.\,1''$$

The quantities in the parentheses are to be taken in seconds.

The answer is in seconds, and additive.

The advantages of the sextant over the theodolite are these:

1. It does not require a fixed support, but can be used while the observer is on horseback, or on a surface in motion, as at sea.

2. It can take simultaneous observations on two moving bodies, as the moon and a star.

It can also do all that the theodolite can. Its only defect is in observing oblique angles in some cases. By these properties it determines distances, heights, time, latitude, longitude, and true meridian, and thus is a portable observatory.

TRILINEAR SURVEYING.

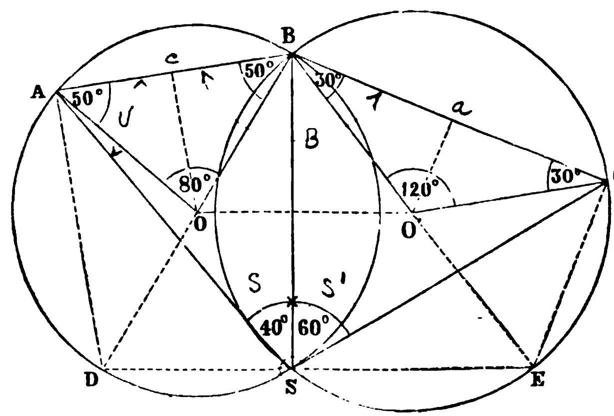
817. Trilinear Surveying is founded on the fifth method of determining the position of a point, by measuring the angles between three lines conceived to pass from the required point to three known points, as illustrated in Art. 8, Part I.

To fix the place of the point from these data is much more difficult than in the preceding methods, and is known as the "Problem of the three points." It will be here solved geometrically, instrumentally, and analytically.

818. Geometrical Solution (Fig. 553). Let A, B, and C be the known objects observed from S, the angles A S B and B S C being there measured. To fix this point, S, on the plot containing A, B, and C, draw lines from A and B, making angles with A B each equal to 90° – A S B. The intersection of these lines at O will be the center of a circle passing through A and B, in the circumference

of which the point S will be situated.* Describe this circle. Also draw lines from B and C, making angles with B C, each equal to $90° - B \hat{S} C$. Their intersection, O′, will be the center of a circle

FIG. 553.



passing through B and C. The point S will lie somewhere in its circumference, and therefore in its intersection with the former circumference. The point is thus determined.

In the figure the observed angles, A S B and B S C, are supposed to have been respectively 40° and 60°. The angles set off are therefore 50° and 30°. The central angles are consequently 80° and 120°, twice the observed angles.

The dotted lines refer to the checks explained in the latter part of this article.

When one of the angles is obtuse, set off its difference from 90° on the opposite side of the line joining the two objects to that on which the point of observation lies.

When the angle A B C is equal to the supplement of the sum of the observed angles, the position of the point will be indeterminate, for the two centers obtained will coincide, and the circle described from this common center will pass through the three points, and any point of the circumference will fulfill the conditions of the problem.

---

* For the arc A B measures the angle A O B at the center, which angle $= 180° - 2\,(90° - A\,S\,B) = 2\,A\,S\,B$. Therefore, any angle inscribed in the circumference and measured by the same arc is equal to A S B.

A third angle, between one of the three points and a fourth point, should always be observed, if possible, and used like the others, to serve as a check.

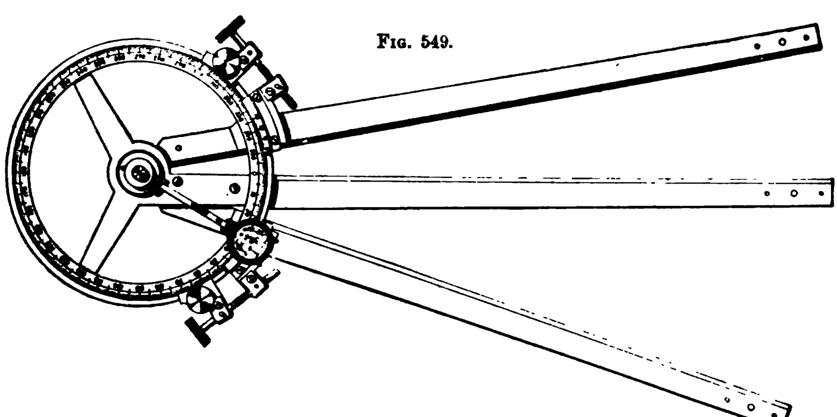
Many tests of the correctness of the position of the point determined may be employed. The simplest one is that the centers of the circles, O and O′, should lie in the perpendiculars drawn through the middle points of the lines A B and B C. Another is that the line B S should be bisected perpendicularly by the line O O′.

A third check is obtained by drawing at A and C perpendiculars to A B and C B, and producing them to meet B O and B O′, produced, in D and E. The line D E should pass through S; for, the angles B S D and B S E being right angles, the lines D S and S E form one straight line.

The figure shows these three checks by its dotted lines.

819. Instrumental Solution. The stations can be more readily found by an instrument called a *station pointer*, or *chorograph*, Fig. 549. It consists of three arms, or straight edges, turning about a common center, and capable of being set so as to make

FIG. 549.



with each other any angles desired. This is effected by means of graduated arcs carried on their ends, or by taking off with their points (as with a pair of dividers) the proper distance from a scale of chords constructed to a radius of their length. Being thus set so as to make the two observed angles, the instrument is laid on a map

containing the three given points, and is turned about till the three edges pass through these points. Then their center is at the place of the station, for the three points there subtend on the paper the angles observed in the field.

A simple and useful substitute is a piece of transparent paper, or ground glass, on which three lines may be drawn at the proper angles and moved about on the paper as before.

320. Analytical Solution. The distances of the required point from each of the known points may be obtained analytically. Let $AB = c$; $BC = a$; $ABC = B$; $ASB = S$; $BSC = S'$. Also, make $T = 360° - S - S' - B$. Let $BAS = U$; $BCS = V$. Then we shall have

$$\text{Cot. } U = \text{cot. } T \left(\frac{c \,.\, \sin. S'}{a \,.\, \sin. S \,.\, \cos. T} + 1 \right)$$

$$V = T \div U$$

$$SB = \frac{c \,.\, \sin. U}{\sin. S}; \text{ or, } = \frac{a \,.\, \sin. V}{\sin. S'}$$

$$SA = \frac{c \,.\, \sin. ABS}{\sin. S}, \quad SC = \frac{a \,.\, \sin. CBS}{\sin. S'}$$

Proof. In the triangle ABS we have

$$\sin. ASB : \sin. BAS :: AB : SB = \frac{AB \,.\, \sin. BAS}{\sin. ASB} = \frac{c \,.\, \sin. U}{\sin. S} \qquad [1.]$$

In the triangle CBS we have

$$\sin. BSC : \sin. BCS :: BC : SB = \frac{BC \,.\, \sin. BCS}{\sin. BSC} = \frac{a \,.\, \sin. V}{\sin. S'} \qquad [2.]$$

Hence $$\frac{c \,.\, \sin. U}{\sin. S} = \frac{a \,.\, \sin. V}{\sin. S'}$$

whence $$c \,.\, \sin. S' \,.\, \sin. U - a \,.\, \sin. S \,.\, \sin. V = 0 \,\ldots\, \qquad [3.]$$

In the quadrilateral ABCS we have

$$BCS = 360° - ASB - BSC - ABC - BAS;$$

or, $$V = 360° - S - S' - B - U.$$

Let $T = 360° - S - S' - B$, and we have $V = T - U \ldots$ [4.]

Substituting this value of V, in equation [3], we get [Trig., Art. 8],

$$c \,.\, \sin. S' \sin. U - a \,.\, \sin. S\,(\sin. T \,.\, \cos. U - \cos. T \,.\, \sin. U) = 0$$

Dividing by sin. U, we get

$$c \,.\, \sin. S' - a \,.\, \sin. S \left(\sin. T \,.\, \frac{\cos. U}{\sin. U} - \cos. T \right) = 0$$

Whence we have

$$\frac{\cos. U}{\sin. U} = \text{cot. } U = \frac{c \,.\, \sin. S' + a \,.\, \sin. S \,.\, \cos. T}{a \,.\, \sin. S \,.\, \sin. T}$$

Separating this expression into two parts, and canceling, we get

$$\text{cot. U} = \frac{c \text{ . sin. S}'}{a \text{ . sin. S . sin. T}} + \frac{\text{cos. T}}{\text{sin. T}}$$

Separating the second member into factors, we get

$$\text{cot. U} = \frac{\text{cos. T}}{\text{sin. T}}\left(\frac{c \text{ . sin. S}'}{a \text{ . sin. S . cos. T}} + 1\right); \text{ or,}$$

$$\text{cot. U} = \text{cot. T}\left(\frac{c \text{ . sin. S}'}{a \text{ . sin. S . cos. T}} + 1\right)$$

Having found U, equation [4] gives V; and either [1] or [2] gives S B; and S A and S C are then given by the familiar "Sine proportion" [Trig., Art. 12].

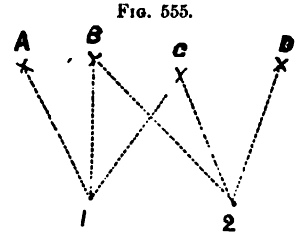
Attention must be given to the algebraic signs of the trigonometrical functions.

Example. A S B = 33° 45′; B S C = 22° 30′; A B = 600 feet; B C = 400 feet; A C = 800 feet. Required the distances and directions of the point S from each of the stations.

In the triangle A B C, the three sides being known, the angle A B C is found to be 104° 28′ 39″. The formula then gives the angle B A S = U = 105° 8′ 10″; whence B C S is found to be 94° 8′ 11″; and S B = 1042·51; S A = 710·193; and S C = 934·291.

821. The High-water Line. The principal points on the high-water line are determined by triangulating. The sections between these points may be surveyed with the compass and chain, by running a series of straight lines so as to follow, approximately, the shore line, and taking offsets from the straight lines of the survey to the bends in the shore line. The straight lines can be more accurately determined by "traversing" with the transit.

Fig. 555.



822. The Low-water Line. In "tidal waters" this is more difficult, because low and bare for only a short time. The survey is best made with the sextant, observing from prominent points to three signals, by the trilinear method, and sketching, by the eye, bends of the shore between the stations observed from.

There should be one to observe and one to record. Let 1 and 2 (Fig. 555) be two points on the low-water line, whose posi-

tion it is desired to determine. The observations taken will be as follows:

1. A and B, 18°.
   B and C, 20°.
2. B and C, 15°.
   C and D, 45°.

When the shore is inaccessible, a base line must be measured on the water, and points on the shore fixed by angles from its ends, as in Art. 827.

823. Measuring a Base on the Water. 1. By sound. Sound travels at the rate of 1,090 feet per second, with the temperature at 30° F. For higher or lower temperatures, add or subtract 1¼ foot for each degree. If the wind blows with or against the movement of the sound, its velocity must be added or subtracted. If it blows obliquely, the correction will be its velocity multiplied by the cosine of the angle which the direction of the wind makes with the direction of the sound.

2. By measuring with the sextant the angular height of the mast of a vessel, then we have

$$\text{Distance} = \text{height of mast} \div \text{tan. of the angle.}$$

824. Soundings. In sounding, the object is to determine the contour of the bottom of any river, lake, bay, etc., so that a chart of it may be drawn, showing the depth of water at all points covered by the survey. The heights of the points on the bottom are referred to the surface of the water as a "datum-plane," and contour lines may be determined in the manner described in *Topography*.

For the same extent of surface, however, if the same degree of accuracy is required, it will be necessary to measure the height of more points in sounding than in topographical surveying, as the surface between the points, whose heights are measured, can not be seen and sketched.

A permanent bench mark must be established, and the height of the water, when the soundings are made, noted and recorded, so that all measurements of depth may be referred to the same datum plane.

For depths up to ten feet a sounding rod, graduated to feet and tenths, may be used. For greater depths, a lead line marked to feet or fathoms, and half fathoms, will be necessary. The size of the line and the weight of the lead will depend upon the depth of the water. A lead weighing ten pounds will be sufficient for depths to forty feet. For greater depths one weighing from fifteen to twenty pounds will be needed. The line should be divided into feet, or fathoms, by leather or cloth tags, and every tenth one made more conspicuous.

325. Signals. An important part of the work on shore before the sounding can begin is the selection of stations and the erection of signals on all the principal points, such as headlands, bights of bays, and in such positions as will be most convenient for use in the work on the water. The position of these signals with respect to the shore survey are determined and mapped.

The important signals may be like those shown in Art. 595.

A good station mark is a post set in the ground about three feet, leaving one foot above the surface. The flag pole is placed in an auger hole made in the top of the post. The flag pole can readily be lifted out, and the transit set over the center of the station. The number of the station should be marked on each post, and it should be distinguished by the combination of colors on the flag, or by the number and arrangement of crosspieces on the staff.

The signals should be made conspicuous by whitewashing or painting.

A pile of stones, or a barrel filled with earth and whitewashed, or a bush with a piece of canvas drawn around it, may be used for signals.

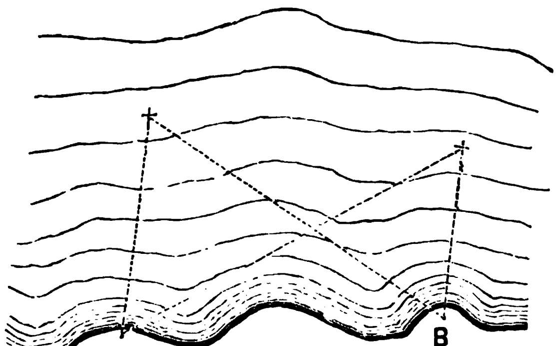
Stations on the water are marked by buoys. A buoy may be made of a light wood float, in which is a hole for the flag pole. The float is anchored with a stone, or by some other means.

Stations may be marked in shallow water in the same way as on the land.

On rocky precipitous coasts signals may be painted on the rocks.

826. Soundings are usually made on parallel lines, at right angles to the shore or to its general direction, and at equal distances or intervals of time. The line on which the boat should go is usually determined by two signals in range, and the points at which

FIG. 556.



soundings are to be taken may be determined by paying out a rope or wire, and directly measuring the distances, or by rowing steadily and taking soundings at equal intervals of time. The latter is the usual method. In this case the whole distance passed over and the time employed must be known, and then the distance between soundings may be readily estimated.

The distance apart of the soundings depends on the regularity of the bottom, the depth of the water, and the object of the survey. Care should be taken to leave no spot unexplored.

The position of the station buoys, and of the boat when sounding, is determined in various ways.

827. From the Shore. A point on the water may be determined by observing to it with a transit from two stations on the shore, at a given signal or fixed time. In Fig. 556, the length of the line A B and the angles which the lines of sight make with it would then be known, and its place would be fixed by angular co-ordinates. Two observers are necessary.

828. From the Boat with a Compass. Observe from the boat with a prismatic compass, or a Burnier's compass, to two signals on

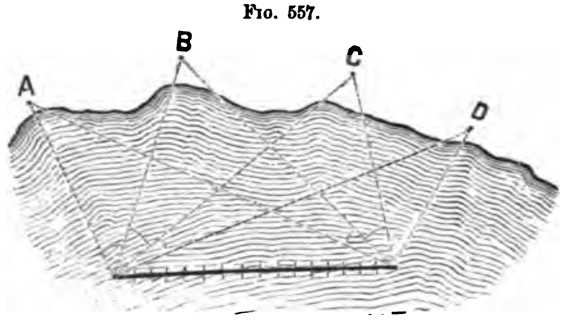
shore. The place of the boat is then determined, and may be fixed on the map by drawing, from the two known points, lines having the *opposite* bearings, and their intersection will be the required point. This is rapid and easy, but not precise.

A modification of this method is to use a range line, given by two signals on the shore, and when the boat is on this line take a bearing, or an angle, to another signal. This fixes the point by the intersection of two lines.

829. From the Boat with the Sextant. Observe with the sextant to three signals on shore, noting the two angles. Two observers, or one observer with two sextants, are necessary. This is the trilinear method, given in Art. 817.

830. Between Stations. It is sometimes necessary to carry on soundings without the use of range lines on the shore. In this case the positions of the boat may be determined by any of the preceding methods at any desired points in the water, and the soundings taken as described in Art. 826; or the distances between soundings may be determined by the use of the "Patent Log," which, when dragged after the boat, registers the distance passed over. The boat may be

Fig. 557.

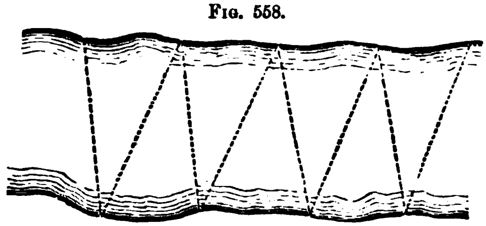


kept in line by the use of the instrument shown in Fig. 111, Part I. In Fig. 557 the two positions of the boat at the ends of the line are determined by the trilinear method.

831. Systems of Lines for Soundings. When great accuracy is desired, besides the lines at right angles to the shore other lines are

run, either at right angles to the first lines, dividing the territory into squares, or crossing the first lines obliquely. This gives an opportunity to check the accuracy of the work by noting the differences of the soundings where the lines cross. On the United States Coast and Geodetic Survey the limit of allowable difference is: For 15 feet and under, two tenths of a foot; between 15 and 30 feet, three tenths; 30 and 48, five tenths; 48 and 72, three fourths of a foot; 72 to 96, one foot and a half; 96 to 150, two feet; and in deep sea, one per cent.

FIG. 558.



For an unimportant water way or bay, or for a reconnaissance, the soundings may be taken in zigzag lines from shore to shore, at equal intervals of time, Fig. 558. In this case it is best to have the alternate courses at right angles to the stream, so as to determine cross sections.

Where soundings can be made through the ice the position of all the points can be determined by any of the methods of surveying. This is the most accurate method of sounding.

832. Tides. On the seacoast the soundings must all be reduced to mean low water, as a datum plane. This necessitates a series of observations on the tides.

Tides are the rising and falling of the water of the ocean twice a day, caused by the attraction of the sun and moon. The tide caused by the sun is much smaller than the tide caused by the moon. When the sun and moon act together the difference between high and low tide is greatest, and these tides are called *spring tides.* When the moon is 90° from the sun the difference between high and low tide is least, and these tides are called *neap tides.* When the water is highest it is called *high water.* When the water is lowest it is called *low water.* *Range* is the height from low to high water.

The horizontal motion of the water caused by the tides is called

the *tidal current*. The tidal current caused by the rising of the tide is called the *flood*, and that caused by the falling of the tide is called *ebb*. *Stand* is the period of time at high or low water when no vertical motion can be detected. *Slack* is the period of time when no horizontal motion can be detected. *Set* is the term applied to the direction of the tidal current, and *drift* to its velocity.

In order to determine the plane of reference with any degree of precision, continuous observations on the tides should be made for at least one month.

833. Tide Gauges. Tidal observations consist in recording the heights of the water at stated times. In order to determine this tide gauges are necessary. The simplest form is a stick of timber graduated to feet and inches, or tenths, and either set up in the water or fastened to the face of a dock or pier, so that the rise of the tide may be noted upon it. The zero point of each gauge is taken at or below the lowest tide, and is referred to a permanent bench mark on the shore. On account of the difficulty of sustaining a timber of considerable height against the force of the wind and waves, several successive gauges are sometimes used—the bottom mark on each gauge higher up being on a level with the top line of the next lower. Such an arrangement is required on gentle slopes.

On the seacoast, where the waves make the reading of the staff difficult, the staff may be attached to a float, inclosed in an upright tube pierced with holes. The holes in the tube should be of such a size as to allow the water to find the mean height inside, and yet reduce the oscillations to very small limits. Permanent tide gauges should be self-registering. For a description of a self-registering tide gauge see "United States Coast Survey Report," 1853.

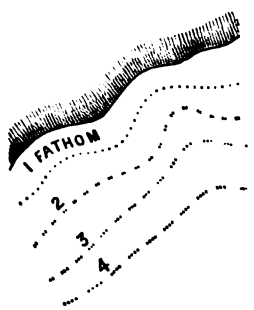
In rivers, a number of tide gauges are necessary, at moderate distances apart, especially at the bends, because the tidal lines of high and low water are not parallel to one another.

The soundings are to be reduced by the nearest gauge, or by the mean of the two between which they may be taken.

834. Beacons and Buoys. Beacons are permanent objects, such as piles of stones with signals on them, usually on shoals and dangerous rocks.

Buoys are floating objects, such as barrels, or hollow iron spheres or cylinders, anchored by a chain, and variously painted, to indicate either dangers or channels.

Fig. 559.



Those placed by the United States Coast Survey are so colored and numbered that, in entering a bay, harbor, or channel, red buoys with even numbers shall be passed on the starboard or right hand, black buoys with odd numbers on the port hand or left hand, and buoys with red and black stripes on either hand. Buoys in channel ways are colored with alternate white and black vertical stripes.

835. The Chart. Having determined the lines of high and low water, the position of the channels, rocks, shoals, etc., and the

Fig. 560.

soundings, a chart must be made, on which all these are laid down in their proper places.

The high-water line is plotted like the bounding lines of a farm. The points determined in the low-water line, and the positions of the boat, determined by the method given in Arts. 827–830, are

fixed on the chart. Contour curves are drawn as in land topography (Chapter XVI) for the first four fathoms. These may be indicated by dotted lines, as in Fig. 559, or they may be shaded with India ink, as in Fig. 560.

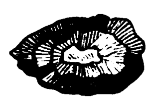
Beyond four fathoms the depths are noted in fathoms and fractions.

Various conventional signs are used (see Art. 801, Topography). Some others are given in the following figures.

FIG. 561.

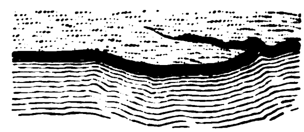
Rocky shore.

FIG. 562.



Rocks always bare.

FIG. 563.



Low, swampy shore.

FIG. 564.

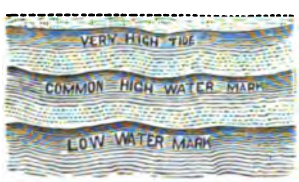
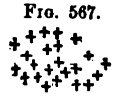


FIG. 565.

Rocks sometimes bare.

FIG. 567.



Reef of rocks.

FIG. 566.

Sandy shore, with hillocks.

FIG. 568.

FIG. 569.

FIG. 570.

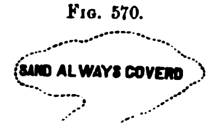


FIG. 571.

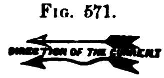
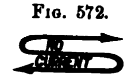


FIG. 572.



CONVENTIONAL SIGNS.

Lighthouse	
Lightship	
Lighthouse (small scale chart)	●
Lighted beacon, stake, or post light	★

Old light tower . . . . ●

Beacon (day beacon, unlighted) . . . . ▲

Spindle, stake, or pile . . . .

No bottom at 20 fathoms . . . .

Kelp . . . . . . ›››

Rock awash . . . . . *

Sunken rock . . . . +

Life-saving station . . +

Wreck . . . . .

Anchorage . . . . ⚓

Buoys (write C., N., or S. for can, nun, or spar, and *whistle* or *bell* for whistling or bell buoy.)

Red (name color, if green, yellow, or white) . .

Black . . . . . .

Horizontal stripes . .

Perpendicular stripes .

Lighted . .

Mooring . . . . .

For details of hydrographical surveying, see "General Instructions for Hydrographic Parties," issued for use in the United States Coast and Geodetic Survey.

CHAPTER XVIII.

UNDERGROUND OR MINING SURVEYING.

836. IT has three objects:

1. To determine the direction and extent of the present workings of a mine.

2. To find a point on the surface of the ground from which to sink a shaft, to meet a desired spot of the underground workings.

3. To direct the underground workings to meet a shaft or any other desired point.

It attains these objects by a combination of surveying and leveling.

SURVEYING AND LEVELING OLD LINES.

837. First Object. To determine the direction and extent of the present workings of a mine.

We have to measure:

1. Azimuths, or directions right and left.

2. Lengths or distances.

3. Heights, or distances up and down, either by perpendicular or by angular leveling; usually the latter.

This being done, the relative positions of all the points are known by their three rectangular co-ordinates.

They are referred, first, to a vertical plane (which may be either north and south, or pass through the first line of the survey); second, to another vertical plane, perpendicular to the preceding one; and, third, to a horizontal datum plane.

In making an underground survey, the same rules and principles apply as to work on the surface. Some differences in methods and detail are necessary, on account of the entire dependence upon artificial light, and the circumscribed limits within which the surveyor is obliged to work.

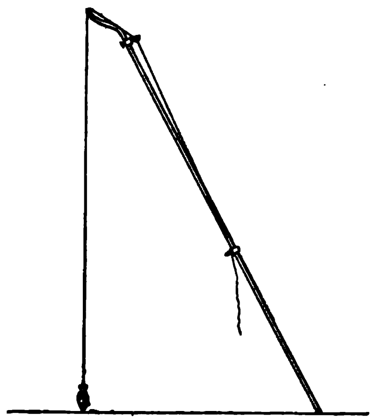
As the headings and airways of a mine are generally driven far in advance of the other workings, and as it is generally more convenient to use the survey stations in the headings and airways as a base from which to extend and check the surveys of the interior portions of the mines, it is essential that they be accurately surveyed and mapped.

838. Stations. The work may often be much simplified by a careful selection of the stations. See that the average distance between them is as long as possible; that they are convenient for future use; and are so chosen that the instrument can be easily set over them. It is also important to locate them where they can be easily and permanently marked. Frequently a station may be so chosen that several different sights can be taken from it, thus economising much time.

839. Marking the Stations. Whenever possible, all stations should be plainly marked with white paint, and given some distinguishing number or letter. This is necessary for use in extending the surveys at some future time, and also to make the map of use when wishing to identify some particular locality in the mine. To avoid confusion arising from duplicate numbers, number the stations consecutively, and continue the same in subsequent surveys. Where different parties are employed to go over the same work, prefix some letter to the numbers for identification. The precise point may be indicated by an iron spud like a horseshoe nail, with a hole through the head large enough to take the line of a plumb bob or plummet lamp. The spud is driven in a crack in the roof, or in a wooden plug which is driven in a hole that has been previously drilled. The objections to this method are, the length of time it takes to get the spuds in the roof, and also the difficulty in using them when the roof is high. Another objection is that mischievous workmen will drive the spuds up in the plugs out of sight with the ends of their drills. Probably as satisfactory a way as any to mark the point is to drill (with a brace and bit in coal) a shallow hole, about one eighth of an inch in diameter, in the center of a painted +, or a circle about six inches in diameter. Fig. 573 shows a very

convenient device for marking the stations, and plumbing down from them when the roof is high. It is made of light gas pipe about half an inch in diameter, and of any convenient length. At one end is a drill; the other end is bent about three inches out of line, and tapered at the end to fit into the hole made with the drill. There is also a notch in the end large enough to hold the line of a plumb bob. Attached to the pipe are two rings with shanks about an inch in length. The lower one is fixed, the other is adjustable with a clamp screw. The upper ring is split in the back wide enough to take a plumb line easily. To use this device in marking the stations, first strike the drill against the roof, then twist it around a few times; this will generally make a mark large enough to be easily identified. Then reverse the instrument, put the handle of the paint brush in the upper ring, adjust to the proper height, and clamp it fast. Put the claw, or notch, in the drill hole and describe a circle, and also paint the number or letter. To plumb down from the point in the roof, remove the brush, put the plumb line in the small notch and through the upper ring, which can be easily done through the spilt. Hold the claw with the plumb line in it against the roof at the proper point, then pay out the plumb line until the plumb bob reaches the bottom, when the point can be fixed. When not in use, bring the two rings together, gripping the plumb bob between them, and clamp fast. Wrap the cord around the shanks of the rings, and fasten with a half hitch.

Fig. 573.



840. Points for setting the Transit over. These may be made in a variety of ways, as a nail in a tie, a chalk × on a rail or stone, a × scratched with a measuring pin, a speck of paint, or a spot of white paint with a speck of coal in the center. If the chalked × is too coarse, rub away a portion of it with the finger. Special cases

may arise where it would be advisable to carry along weights of lead with a short piece of brass wire projecting above the surface, to give a precise point. A center mark on the top of the telescope will afford the means of placing the transit in position under a plumb bob suspended from the roof.

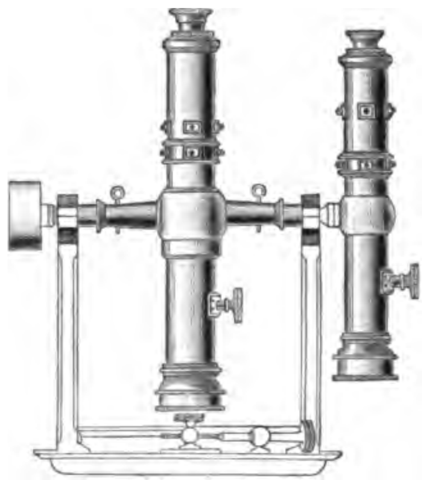
841. Giving the Sights. A measuring pin, if held plumb, with a lamp in front and a little to one side, makes a very good sight. The pin should be whitened with chalk, to make a background for the cross hair. The cord of a plumb bob can be seen distinctly up to three or four hundred feet, if a piece of white paper is held behind it and a light is held in front. Care must be taken not to mistake the shadow of the line for the line itself. It is difficult to hold the plumb bob steady unless it can be hung in the iron spuds mentioned in Art. 839, or the device shown in Fig. 573 is used. Where the mine is smoky, or the sights are very long, sight to the center of the blaze of the lamp, which must be carefully plumbed over the point. To meet cases of this kind, the plummet lamp has been devised (Fig. 574). It consists of a brass lamp hung in gimbals and supported by two chains. The lamp terminates below in a conical plummet. A shield at the top prevents the flame from burning the string. The sight is taken to the center of the flame. These lamps are generally used in pairs, for back and forward sights. They are inconvenient to use, as they require the iron spuds with a hole through the head to support them from the top. Where the roof is high, it is difficult to get up to the station to put the string through the hole. If care is taken not to make them too heavy, they can be supported with the device mentioned in Art. 839. Another objection is the additional load they impose upon the party to carry.

Fig. 574.

842. The Transit. The essential features of a transit to be used for surveys in mines are that the verniers should be so placed as to

be easily read by lamplight, and that the marking should be very distinct, on account of the imperfect light available. Again, the instrument should not be too heavy, as there is often difficult climbing to be done over fallen rock and other mine *débris*. If the instrument be easily detached from its tripod, it will often be found a convenience, as thereby the load may be lightened and the instrument itself more carefully carried and more fully protected.

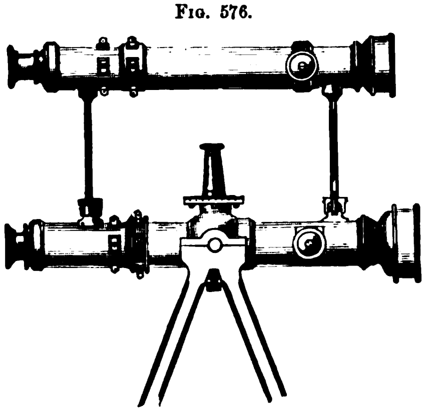
Fig. 575.



Graduations on solid silver are apt to be tarnished by the powder smoke of the mines. Some makers claim to obviate this by making the graduations on platinum.

If the telescope has a level attached, see that the lamp is not held under it for any length of time, as the heat may explode it. Accidents of this kind have occurred, producing serious results.

Fig. 576.



In one form of mining transit an extra telescope is attached on one side, as shown in Fig. 575, and is balanced by a weight on the opposite side. The advantage of this form is, that sights may be taken vertically up or down, as is sometimes necessary in connecting the underground surveys with those on the surface.

In another form, the extra telescope is attached to the transit telescope, as shown in Fig. 576, or as in Fig. 344, Part I.

The diagonal prism, shown in Fig. 211, Part I, may be used with advantage on the extra telescope.

843. Taking the Sights. The beginner will at first have some trouble in catching the light through the telescope. A little practice will overcome this. Hold a lamp a little above the instrument, sight over the top of the telescope, and turn it until it points to the light which it is desired to observe. Now sight through the telescope, and turn it a little each way, until the eye catches the light. Clamp the instrument, and move the object glass until the light looks like a large round blur. This will form a background on which the cross hairs can be plainly seen. Bisect the blur, then focus the object glass, and the cross hairs will be so near the right place that there will be no trouble to find them in bisecting a plumb line, or whatever else is sighted to. Some instruments have a reflector for illuminating the cross hairs by throwing a light into the telescope (Fig. 210, Part I). The same result can be accomplished by holding a lamp two or three feet in front of the object glass, and a little to one side, so as to be out of the line of sight.

844. Measuring the Angles. Proceed as in making a traverse on the surface, noting whether the angles are to the right or left. It is generally more satisfactory to put the vernier at zero every time rather than to survey or traverse by the back angle. The instrument gets some hard usage, and when the surveyor reviews the angle, after having moved to the next station preparatory to measuring a new angle, he has the unsatisfied feeling of not knowing whether the upper motion has slipped, or that he read the angle wrong before. It is also more troublesome to set the vernier at odd degrees and minutes than at 0, in case there should be a slip of the upper motion. The surveyor should never omit to check the reading of his angles, either by noting whether the sum of the two readings on each side of the 0 of the vernier is equal to 180° or by repeating the angle. The latter method is the most satisfactory. If the graduated circle has a double row of figures reading 180° each way, and the deflection should be greater than 90°,

it is only necessary to read the supplement or smaller angle, noting at the same time whether it reads to the right or left on the limb.

The needle readings, which should always be taken, will prevent the gross error of getting into the wrong quadrant.

	BACKSIGHTS.	ANGLES.	FORESIGHTS.	
Thus,	S. 30° 00′ W.	165° 00′ L	N. 45° 00′ E.	is the same
as	S. 30° 00′ W.	15° 00′ R	N. 45° 00′ E.	the needle,

showing that the last course should be N. E. instead of S. W., as the angle would seem to indicate.

The advantage of this method is that it is a little more convenient to use in working out the courses. It also relieves the surveyor of the inquiry as to whether his vernier has passed the 90°, and he should use the larger or smaller angle. He reads the vernier as it stands, and lets the needle determine the quadrant.

There is the objection, however, that, in deflecting very small angles near the quadrant points, an error in noting whether the angle is right or left can not be detected by the needle. It is preferable to have the circle graduated with a double row of figures to 360° each way, and to measure the angles without reversing the telescope. Errors of adjustment are thus eliminated, and direction given without any regard to quadrant or needle reading. It is almost impossible to set up an instrument so solidly that when the cross hairs are put on a given point they will remain there for any length of time. For this reason it is best not to begin to measure the angle until everything is all ready; then measure and check by doubling it as quickly as can be done with accuracy. Occasions sometimes arise in which a surveyor has but a few hours in which to make an extended survey. For a necessity of this kind the use of three transits will be found to expedite the work very greatly. This prevents loss of time in setting the instrument over a given point, the work being carried on from the plumb line of one instrument to that of the next.

345. Plumbing the Shaft. In order that the lines underground may be worked from the same meridian as those on the surface, they must be deflected from some line whose azimuth is known. Should it not be considered justifiable to depend upon the needle to determine the azimuth, and should it be impossible to enter the mine by a *slope* or a tunnel, the surveyor will be obliged to resort to plumbing the shaft. Two plumb lines are carefully put into some known line on the surface, and their direction, which will be in the same line, is again taken at the foot of the shaft, as a meridian from which all the lines underground are deflected. As the two plumb lines are necessarily but a few feet apart, and as the integrity of all the subsequent work depends upon the accuracy with which the direction of the line on the surface is reproduced by the plumb lines at the foot of the shaft, it is necessary that extreme care should be exercised in doing the work. Much time will be saved by studying the local conditions of the shaft, and making thorough preparations before beginning the work. In the selection of wires, iron and steel are excellent, when new, as their strength enables a fine wire to support a heavy weight. The objection is that they rust and become treacherous, breaking at most inopportune times. Hard-rolled brass wire, though free from this objection, has to be very carefully used, as it is liable to kink, and then break. If it slips out of the hands while attaching the weights at the bottom, it will fly up the shaft in an almost inextricable tangle. Copper stretches, and the weights have to be carefully watched to see that they do not touch the bottom of the vessel in which they are suspended. On the whole, however, it seems to give the best satisfaction. Have the wire wound on two strong reels, set in frames which can be securely anchored. The reels should have stops, so that the weights can be held at any point that may be desired.

Suspending the Wires.—Nail two boards on the sides of the head frame, at right angles to the line of sight, and about four feet from the ground. Place on each of these boards a scantling about twelve feet long, letting one end rest on the ground a little out of the line of sight. The upper end should project over the shaft far enough to clear the sides. Put the reels in position, about twenty feet back from the shaft and also a little out of the line

of sight, and anchor them securely. Fasten weights of about five pounds each to the ends of the wires, and pass them over the ends of the scantlings. Then pay out the wires until the bottom of the shaft is reached. Bring the wires approximately into line by tapping the scantlings with a hammer. In the meantime the assistants at the foot of the shaft will attach the weights of twenty-five or thirty pounds and place them in pails of water. When the signal is given that all is right below, the wires are brought precisely into line, putting in the wire farthest from the instrument first, then bringing the other to it. This can be very easily and accurately done by tapping the scantling gently with a hammer. Examine the wires from the top to the bottom of the shaft to be sure they touch no projecting points. Make all secure at the surface, and, before taking up the instrument to go below, review the work, to be sure that all is correct. Be very careful that no work is done over the head of the shaft while men are at work in the shaft at the foot, lest accidents should occur. At the bottom of the shaft nail two boards across the foot frame, the same as at the surface. On these place two other boards about ten inches wide and one quarter of an inch apart, and reaching across the shaft so that the wires will swing freely in the crack between them. These boards serve as a rest for the hand in steadying the vibrations of the wires. They also prevent drops of water from falling into the pails and producing currents which will move the weights. Take a small piece of board and bevel one edge slightly with a knife; then lay it across the crack between the boards, and bring the beveled edge slowly up to one of the wires until it almost touches. Make a mark on the edge where it bisects the wire, then watch to see if the wire is perfectly still. In deep shafts the oscillations of the wire are very slow, and it is trying to the eye to watch them through the telescope until they are perfectly still.

Sometimes wires may be steadied by uniting them with a thread or string slightly shorter than the distance between them. The weights are also made with radiating arms or wings to increase their resisting surface, and sometimes placed in oil or mercury. Molasses has also been suggested. If it is impossible to perfectly steady the wires, fasten them at the mean of the oscillations.

Getting the instrument into line is not an easy task for the beginner, owing to the difficulty in distinguishing between the lines when looking through the telescope. This is overcome by an assistant holding a white paper with a light alternately in front of and behind the wire farthest away. Another method is to put a couple of round rings in the first wire, and then the second wire can be seen through the openings in the rings. Another very good way is to tack a piece of sheet iron, of about eight by ten inches, to a piece of board of the same size. Make a hole about one sixteenth of an inch in diameter in the center of the sheet iron, and at the height of the center of the blaze of a mine lamp above the board. Bend the sheet iron so that it will be slightly convex with the bend at the hole. Place this contrivance behind, and as close as possible to the rear wire, with the small hole bisecting it. Place a lighted lamp behind the sheet iron so that the blaze will cover the hole. Put a small piece of board with white paper tacked on it behind the first wire; also a lighted lamp in front. The instrument can now very quickly be brought into line with the first wire, and the point of light at the second. Verify by holding white paper, with a light, behind the second wire, and noting whether it is entirely concealed by the other wire.

If possible, use two transits, placed on opposite sides of the shaft, then verify by seeing if they bisect each other's plumb lines. Do not try to set up the instrument too far away, as it increases the difficulty of getting a clear sight of the wires. Watch, also, that the shadow of the wire is not mistaken for the wire itself. When all is completed, mark the line permanently for future use. Where great accuracy is required, plumb the shaft several times, and take the mean, depending also upon which of the several plumbings has been done with the least probability of error.

Second Method.—When there are two shafts convenient to each other, let a plumb line down each shaft; then connect them by a careful survey, both on the surface and underground. Calculate the course between the lines on the surface. Calculate also the course between the wires underground from an assumed meridian. The difference between the two courses will be the correction to be

applied to the underground courses to make them correspond with the azimuth assumed on the surface.

Third Method.—Use a transit with a telescope outside the standards, Figs. 575 and 576. Place the instrument in line directly over the shaft, then produce the line to the foot of the shaft by revolving the telescope so as to sight directly down the shaft. Get two points as far apart as possible at the foot of the shaft, then stretch a fine wire carefully over them, producing the line far enough to make a convenient station over which the transit can be set. In shallow shafts, where communication between the top and bottom is easy, the wire may be lined in directly with the instrument.

Fourth Method.—If no local attraction exists, and extreme accuracy is not required, use the needle. The needle can be read to within five minutes, and the errors have the probability of correcting each other in the different courses taken. If there is only time and means to do ordinary work, it is better to depend exclusively upon the needle than upon plumbing and deflections poorly done.

The beginner should remember that the greatest care is necessary, and that, when his best has been done, there are possibilities of error. A surveyor who appreciates these errors will not fail to verify his work by repetitions at a later date; as, by making a connection with other openings to the surface, such as a drill hole, an opening for air, or a connection through a neighboring mine, should such an opportunity present itself.

846. Keeping the Notes. These will depend very much upon the character of the work to be done. Some surveyors prefer to use two notebooks. In one are recorded all the instrumental work done with the transit, together with the stations, and such explanatory remarks as may be necessary. In another, made especially for the purpose, are kept all measurements and references, accompanied with a sketch showing where they were taken. Where the party is large enough, it may be divided so that both of these kinds of work may be kept going at the same time. Another method, much used, is to keep all the work in one book, where everything will be all together when it is wanted. By having the

March 4, 1886.—Near Foot of Shaft 14.

Set up on 51. B. S. on 52.

Station.	Positive Angles.	Negative Angles.	Bearings.	Distances.	Slope ±.	Height to Roof.
	° ′	° ′	° ′		° ′	
51-70.	180-00	180-00	N. 55-30 W. N. 56-50 W.	39·2	−0-45	Rail. 7·42
70-71.	39-45	220-15	S. 84-15 W. S. 83-00 W.	121·0	+2-05	Rail. 10·25
71-72.	202-06	57-54	N. 73-39 W. N. 72-10 W.	126·0	+0-55	Pave. 9 73
72-⊕ 93.	84-25	275-35	S. 10-46 W. S. 12-15 W.	93·0	+7-35	Pave. 5·21
72-⊕ 104·3.	167-31	192-29	N. 86-08 W. N. 86-05 W.	104·3	+1-02	Pave. 11·43
⊕ 104·3-V. 1.	105-33	254-27	S. 19-25 W. S. 19-30 W.	84·5	+10-02	Tie. 4·23
⊕ 104·3-74.	270-05	89-55	N. 3-57 E. N. 4-00 E.	41·8	−3-01	Tie. 6·75
74-75.	271-21	88-39	S. 84-42 E. S. 84-40 E.	78·3	−0-32	Tie. 7·21
75-76.	190-09	169-51	S. 74-33 E. S. 74-25 E.	125·7	−0-22	Rail. 7·35
76-77.	163-01	196-59	N. 88-28 E. N. 89-00 E.	144·9	−0-08	Pave. 14·12
77-40.	176-53	183-07	N. 85-21 E. N. 89-55 E.	217·0	−0-15	Pave. 7·52
77-H.	43-17	316-43	N. 48-15 W. N. 50-30 W.	73·6	−4-12	Pave. 6·25
77-50.	213-34	146-26	S. 57-58 E. S. 53-35 E.	99·3	−0-30	Rock. 7·15
50-49.	142-44	217-16	N. 84-46 E. N. 89-10 E.	N. 84-47 E. Error 0-01.		

Begin at V. 1 above to run short chambers.

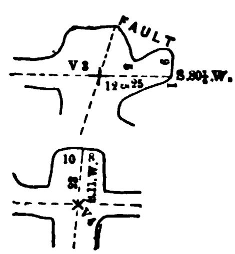
V. 1-V. 2.			S. 80-45 E.	36·4	−2-01	Pave. 4·92
V. 2-V. 3.	261-44	98-16	S. 1-00 W.	20·0	+15-08	Pave. 5·23
V. 3-V. 5.			N. 85-15 E.	79·8	−6-30	Pave. 5·21
V. 5-D. 1.	278-36	81-24	S. 3-50 W.	43·0	+25-00	Rail. 8·20

Set up on ⊕ 53 on line between 74 and 75. B. S. on 74.

74-⊕ 53.			S. 84-42 E. S. 84-40 E.			
⊕ 53-N.	99-57	260-03	N. 15-15 E. N. 15-15 E.	77·2	−10-13	8·7

R. = right. ⊕ = mark for future use. ○ = width of place is put in circle.
L. = left.] = face and stopped. P. = pillar.

	18, door $\frac{21\text{ P. R.}}{10\cdot5}$ $\frac{89\cdot2}{8\cdot11}$
$-\frac{12}{4-}-\frac{20\text{ P. R.}}{7-}-$	$\frac{25}{-2}-50\ \oplus\ \}\frac{8}{58\text{ R.}}$ $\frac{8}{119\text{ R.}}$ $\frac{0}{8\cdot11}$ $\frac{25}{8\cdot8}$ $\frac{50}{1\cdot10}$ $\frac{100}{9\cdot8}$ $\frac{\text{dist.}}{6\cdot2}$
	$\frac{15}{4\cdot6}$ $\frac{84\text{ P. L.}}{8-8}$ $\frac{50}{2-}$ $\frac{60}{6-}$ $\frac{77\text{ F. R. L.}}{5\cdot6}$ $\frac{80\cdot2\ \oplus}{5\cdot6}$ $\frac{81\text{ P. R.}}{5\cdot6}$ $\frac{100}{4\cdot7}$ $\frac{\text{dist.}}{5-}$
11 40 L. 10 50 R. 16 95 R.	$\frac{7\text{ P. R.}}{8\cdot10}$ $\frac{40}{10\cdot10}$ $\frac{60}{10\cdot16}$ $\frac{75}{10\cdot18}$ $\frac{98\ \oplus}{10\cdot12}$
$\frac{8}{80\text{ R.}}$ $\frac{7}{102\text{ R.}}$	57, door. $\frac{20}{8\cdot8}$ 30 $\frac{80\text{ ch. L.}}{5-7}$ $\frac{50}{9\cdot2}$ 23 $\frac{65\text{ ch. L.}}{8-5}$ $\frac{100}{8\cdot2}$ 18 $\frac{108\text{ ch. L.}}{10\cdot8}$ $\frac{124\cdot8}{7-8}$ P. ⊕ 78 + 10 face]
10 20 R. 10 50 R. $\frac{8}{48\text{ L.}}$	$\frac{8\text{ P. L.}}{15\cdot8}$ $\frac{20}{12\cdot5}$ $\frac{48}{12\cdot10}$ $\frac{88}{10-}$ +10 face. Up ch.
	10 rb. $\frac{82\text{ N. R.}}{4\cdot8}$ +6 F. R.
$-\frac{18}{12\cdot4}$ face] }	58 ⊕ $\frac{7}{70\text{ R.}}$ $\frac{25}{10\cdot6}$ $\frac{50\text{ P. L.}}{8\cdot8}$ $\frac{\text{dist.}}{9\cdot6}$
	$\frac{25}{11\cdot8}$ $\frac{50}{10\cdot6}$ $\frac{65}{12\cdot2}$ $\frac{80}{22\cdot4}$ $\frac{85\text{ P. R.}}{8\cdot6}$ $\frac{100}{9\cdot7}$ $\frac{\text{dist.}}{7\cdot7}$
$\frac{10}{15\text{ R.}}$ $\frac{8}{7\cdot8\text{ R.}}$	$\frac{20}{8\cdot4}$ $\frac{50}{10\cdot8}$ $\frac{75}{18\cdot8}$ $\frac{100}{14\cdot2}$ $\frac{120}{15\cdot1}$ $\frac{\text{dist.}}{14\cdot8}$ $\frac{188\text{ P. L.}}{15\cdot8}$
Close on 40.	$\frac{15}{12\cdot5}$ $\frac{25\text{ P. R.}}{6\cdot5}$ $\frac{50}{8\cdot8}$ $\frac{70}{10\cdot8}$ $\frac{100}{9\cdot8}$ $\frac{125}{10\cdot8}$ $\frac{150}{10\cdot8}$ $\frac{170}{11\cdot4}$ $\frac{200}{9\cdot7}$ $\frac{207}{7\cdot7}$ $\frac{\text{dist.}}{10\cdot8}$
	$\frac{5\text{ P. L.}}{12\cdot1}$ $\frac{50}{10\cdot8}$ $\frac{78}{8\cdot7}$ +10 stopped by water.
Close on 50.	$\frac{10}{6-}$ $\frac{15}{2-}$ $\frac{25\text{ door \& P. R. \& L.}}{2-9}$ $\frac{85}{-10}$ 56·9 to ⊕ 50 Course 1. $\frac{50}{4\cdot8}\cdot\frac{75}{8\cdot5}\cdot\frac{85}{8\cdot8}\cdot\frac{97}{8\cdot11}$.
Close on 49.	
	$\frac{10\text{ rb.}}{12\cdot5}$ $\frac{25\text{ N. R.}}{-7}$ +11 fr. rb. $\frac{20}{12\cdot5}$
12 L.	$\frac{12}{15\cdot12}$ $\frac{80}{10\cdot17}$ $\frac{40}{8\ 18}$ $\frac{50}{0\cdot2}$ face
Close on ⊕ 98, top of page. }	$\frac{18\text{ rb.}}{8\cdot10}$ $\frac{44\text{ V. 4}}{\text{face 28 R.}}$] $\frac{78\text{ rb.}}{8\cdot7}$ +12 fr. rb.
Close on D. 1, Evan Jones's cha. }	$\frac{10}{5\cdot11}$ $\frac{15}{8\cdot8}$ $\frac{25}{8\cdot8}$
	$\frac{10\text{ rb.}}{5\cdot4}$ $\frac{90}{7\cdot4}$ $\frac{80}{7\cdot8}$ $\frac{50}{8\cdot8}$ $\frac{\text{dist.}}{8\cdot8}$ +7 face.



F. R. = far rib.	⌐ = blind entrance.	rb. = rib.	hdg. = heading.
N. R. = near rib.	dist. = distance.	ch. = chamber.	Pave. = pavement.

figures represent certain things when in particular places, and the use of a few symbols and small sketches in special localities, a notebook kept in this manner can generally be made to convey all needed information. Below will be found the right- and left-hand pages of a notebook kept in this manner; also a map showing the portion of the mine included in the survey of which the notes are a part.

In the first column are the station numbers, which correspond with the marks painted in the mine. The brackets indicate that several sights were taken from one station.

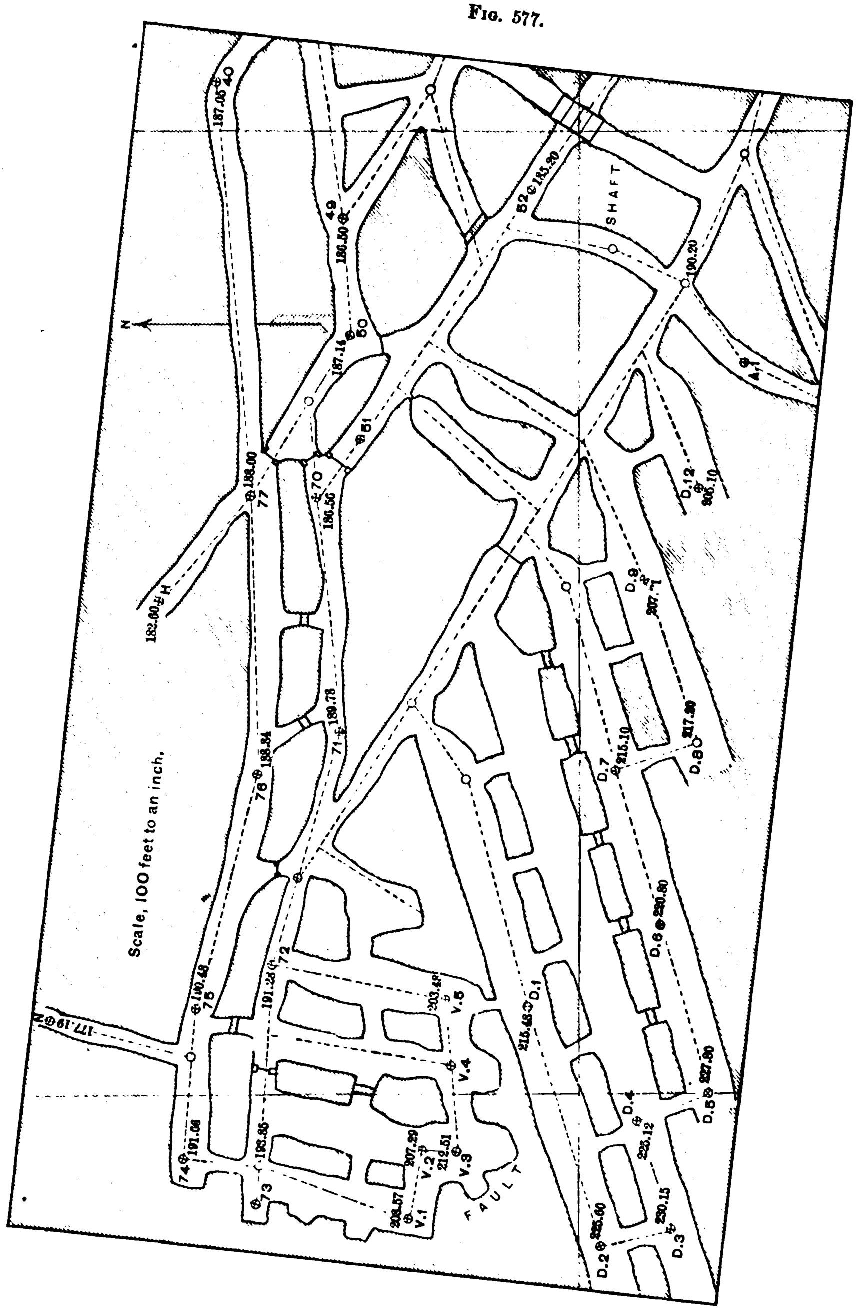
In the second and third columns are recorded the right and left angles taken at each station, the sum of which should equal 360°. These angles are measured without reversing the telescope; in this particular case a transit being used which had a double row of figures reading to 360° each way. The fourth column, the needle courses with the corrected courses placed above them in red ink. Fifth column, distances. Sixth column, slopes, and whether plus or minus.

Where the conditions are such that the line of sight can not be taken parallel with the roof or bottom, then additional room must be taken to record height of instrument, as also height of object sighted to above bottom or below roof. Seventh column, height to roof. On the right-hand page, measurements from station 70 to 71 would be called out by the chainman as follows: Produce 70 and 71 back, indicated by the minus signs before the distances preceding the bracket. At − 12, 4 right; at − 20, first end of pillar 7 right; at − 25, 2 left; at − 50 leave a point for future reference; at 0, or station 70, 5 right and 9 left; at 25, 3 right and 8 left; at 58, 1 right and 10 left; at 58, entrance right, 8 wide and walled; at 100, 9 right and 3 left; at 119, entrance right, 8 wide and walled; at distance, or station 71, 8 right and 2 left, etc.

There will occur to the surveyor in practice various symbols and abbreviations which he can use to lessen the labor of recording.

847. Tabling the Survey. On pages 416 and 417 will be found a form and the tabling of the above field notes for office use and record. It is best to have a specially prepared book already ruled

Fig. 577.



STATIONS.	COURSE.	DISTANCE.	SLOPE + OR —.	SLOPE DISTANCES REDUCED. 1st.	SLOPE DISTANCES REDUCED. 2d.	COURSE AND DISTANCE REDUCED. 1st.	COURSE AND DISTANCE REDUCED. 2d.
70............	° ′ N. 55·30 W.	*39·19* 89·20	° ′ —0·45	88·99 ·20 89·19	0·51 ·00 —0·51	82·14 ·16 82·30	22·09 ·11 22·20
71............	S. 84·15 W.	*120·96* 121·00	+1·31	99·97 20·99 120·96	2·66 ·56 +3·22	119·40 ·96 120·36	12·02 ·09 12·11
72............	N. 73 39 W.	*125·99* 126·00	+0·41	99·99 26·00 125·99	1·19 ·31 +1·50	95·96 23·99 ·95 120·90	28·15 7·04 ·28 35·47
73 is 20 beyond station......	N. 86 08 W.	*104·28* 104·30	+1–02	99·96 4·30 104·28	1·80 ·77 +2·57	99·77 3·99 ·28 104·04	6·74 ·27 ·02 7·03
74............	N. 3 57 E.	*41·74* 41·80	—3·01	40·94 ·80 41·74	2·15 ·04 —2·19	40·90 ·78 41·68	2·82 ·05 2·87
75............	S. 84 42 E.	*78·30* 78·30	—0 52	78·00 ·30 78·30	1·18 ·00 —1·18	77·67 ·30 77·97	7·20 ·08 7·28
76............	S. 74 33 E.	*125·70* 125·70	—0 45	120·00 5·70 125·70	1·57 ·07 —1 64	115·65 5·49 121·14	31·97 1·51 33·48
77............	N. 88 28 E.	*144·90* 144·90	—0 20	140·00 4·90 144·90	0·82 ·02 —0·84	139·95 4·90 144·85	3·75 ·12 3·87
Point on line between 77 & 50	S. 57–58 E.	*56·89* 56·90	—1 00	55·99 ·90 56·89	0·98 ·00 —0·98	47·47 ·75 48·22	29·70 ·47 30·17
Close on 70....	S. 84 15 W.	*50·00* 50·00	—0 32	50·00	—0·46	49·75	5·01
From 72 to V. 5.	S. 10 46 W.	*91·65* 92·46	+7·35	91·20 ·45 91·65	12·14 ·06 + 12 20	89·40 ·64 90·04	17·00 ·18 17·18
From 20 back of 73 to V. 1....	S. 19 25 W.	*82·21* 84·50	+10 02	82·72 ·49 83·21	14·63 ·09 +14·72	78·28 ·20 78·48	27·59 ·07 27·66
Old sta. From 77 to 40...	N. 85 21 E.	*216·99* 217·00	—0 15	209·99 7·00 216·99	·92 ·03 —0·95	209·31 6·97 216·28	17·02 ·57 17·59
From 77 to H...	N. 48 15 W.	*73·49* 73·68	—4 12	72·81 ·68 73·49	5·35 ·05 —5·40	54·46 ·37 54·83	48·61 ·33 48·94
Old sta. From 77 to 50...	S. 57 58 E.	*99·30* 99·30	—0 30	99·00 ·30 99·30	·86 — ·86	83·93 ·25 84·18	52·51 ·16 52·67
V. 2......	S. 80 45 E.	*36·38* 36·40	—2 01	35·98 ·40 36·38	1·27 ·01 —1·28	35·53 ·37 35·90	5·79 ·06 5·85
V. 3...... (Not deflected.)	S. 1 00 W.	*19·30* 20·00	+15·08	19·30	+5 22	19·00 ·30 19·30	0·33 ·01 0·34
V. 5......	N. 85 15 E.	*79·28* 79·80	— 6 30	78·49 ·79 79·29	8·94 ·09 —9·03	78·73 ·28 79·01	6·54 ·02 6·56
Close on D. 1.	S. 3 50 W.	*43·00* 44·70	+15 46	42·33 ·67 43·00	11·96 ·19 +12·15	42·90	2·87
From 74 to ⊕ 53.	S. 84–42 E.	53·00	—0·52	53·00	—0·80	52·77	4·90
From ⊕ 53 to N.	N. 15–15 E.	*75·97* 77·20	—10·12	75·78 ·19 75 97	18·63 ·04 —18·67	72·86 ·94 73·80	19·73 ·25 19·96

N.	S.	E.	W.	ALGEBRAIC SUM OF LATITUDES.	ALGEBRAIC SUM OF DEPARTURES.	ALGEBRAIC SUM OF SLOPES.	HEIGHT OF ROOF.
				+112·00	−159·00	+187·07	See sta.50
22·20			82·30	+134·20	−191·30	+186·56	Rail. 7·42
	12·11		120·36	+122·09	−311·66	+169·78	Rail. 10·25
35·47			120·90	+157·56	−432·56	+191·28	Pave. 9·78
7·03			104·04	+164·59	−536·60	+193·85	Pave. 11·48
41·63		2·87		+206·22	−533·73	+191·66	Tie. 6·75
	7·23	77·97		+198·99	−455·76	+190·48	Tie. 7·21
	33·48	121·14		+165·51	−334·62	+188·84	Rail. 7·35
3·87		144·85		+169·38	−189·77	+188·00	Pave. 14·12
	30·17	48·22		+139·21	−141·55	+187·02	
	5·01		49·75	+134·20	−191·30	+186·56	
88·00	88·00	395·05	395·05				
	90·04		17·13	+67·52	−449·69	+203·48	Pave. 5·21
	78·48		27·66	+86·11	−564·26	+208·57	Rock. 4·23
17·59		216·28		+186·97	+26·51	+187·05	Pave. 7·52
48·83			48·94	+218·42	−244·60	+182·60	Pave. 6·25
	52·67	84·18		+116·71	−105·59	+187·14	Rock. 7·15
	5·85	35·90		+80·26	−528·36	+207·29	Pave. 4·92
	19·30		0·34	+60·96	−528·70	+212·51	Pave. 5·23
6·56		79·01		+67·52	−449·69	+203·48	Pave. 5·21
	42·90		2·87	+24·62	−452·56	+215·63	Rail. 8·20
	4·90	52·77		−201·30	−480·96	+190·86	Pave. 7·50
73·30		19·98		+274·60	−460·98	+177·19	Rail. 8·45

to the required form. All the work of tabling can then be done in this book. Should there ever be an occasion to review the work, it can easily be found.

The two double columns headed 1 and 2 are for convenience in taking down the numbers as they are called off from Gurden's "Traverse Tables," which are to single minutes, and distances to one hundred feet. For convenience in description, we will suppose two persons, A and B, to be tabling the above survey. A will take the sheet on which have been recorded the stations, corrected courses, distances, and slopes, and call out the angle, which in the present case we will suppose to be N. 55° 30′ W., distance 39·19. B finds this in the book of tables, and on the edge of a sheet of blank paper checks the heavy line on the center of the page; also, the two minute columns. A then calls out the distance, 39·19, which B sets down on his sheet of paper, and then, using his paper as a straightedge, slides it down the page until he comes to 39, taking care to keep the check on the center line. He will then call out the numbers under the checks for the minute columns, *always* reading the left-hand one first, to A, who will record them as he receives them in columns 1 and 2. The same operation is repeated for the 19. A will then call out the next angle, and while B is searching for it he will add the numbers given, and, if he has time, carry the results out to the proper columns of N., S., E., and W. A glance at the course, noting whether it is greater or less than 45°, will tell him whether the larger number should be put in the column of Latitude or Departure. The same operation is repeated for all the courses.

For convenience in plotting and calculations, the latitudes and departures should all be referred to a common origin of co-ordinates. In this survey the origin is taken at the west plumb line of the shaft. Station 51 has been found by previous work to have latitude north + 112, and departure west 159. In like manner, 51 has been found to have a + elevation of 187·70. The slopes and distances should be reduced first, then the corrected horizontal distances placed over the others in red ink.

Problem.—It is desired to drive the heading from H so that it will intersect the slope at N. Required the course and distance.

From the columns of total latitudes and departures in the sheet of calculations take

	Latitude.	Departure.
N =	+ 274·60	− 460·98
H =	+ 218·42	− 244·60
	+ 56·18	− 216·38

Tangent, of course, equal departure divided by latitude.

$$\begin{aligned}
\log. 216{\cdot}38 &= 2{\cdot}3352171 \\
\log. 56{\cdot}18 &= 1{\cdot}7495817 \\
\text{tem. } 75 - 27 &= 10{\cdot}5856354 = 75^\circ\ 27' = \text{course} \\
\log. 56{\cdot}18 &= 1{\cdot}7495817 \\
\cos. 75^\circ\ 27' &= \bar{9}{\cdot}4000625 \\
& \quad 2{\cdot}3495192 = 223{\cdot}62 = \text{distance}
\end{aligned}$$

N being north and west of H, shows the course to be N. W., or N. 75° 26′ W.

Unless in special cases, where great accuracy is required, the more common method of solving this and similar problems is to take the course and distance from the map with a protractor and scale, this being sufficiently accurate for all practical purposes.

848. Making the Map. If the map is to be much handled, use the best quality of cloth-backed paper. The edges should be bound with linen tape, which, if sewed, should be double stitched, with about three stitches to the inch. If the stitches are made closer than this the binding will break off in the line of the needle holes. Ascertain from existing map, or whatever data may be at hand, the most advantageous direction for the meridian of the survey to assume on the map. Fix also upon a point for the origin of co-ordinates. Begin at the origin, and rule the paper into five- or ten-inch squares parallel with the meridian of the survey. Very great care is required in doing this work, in order to make all the squares check precisely with the scale and be rectangular. Owing to the expansion and contraction of the paper, the work of laying out the squares should be concluded on the same day it is started. The map should show all land lines, dwellings, roads, streams, ponds of water, and any other features that may have a bearing on an intelligent working of the mine. Both surveys should be referred to

the same origin of co-ordinates. In plotting an underground traverse, it is generally more convenient to locate only every fifth or tenth station by its co-ordinates, and use a protractor for filling in the balance.

Take a paper protractor and letter it N. S. E. W., and fix it at any convenient place on the paper, so that its N. and S. points will correspond with the meridian of the survey. Fasten with weights; then transfer the courses from the protractor to where they are wanted on the map, scaling off the distances as required. The stations that have been located by ordinates will check the slight errors in the plotting from the protractor. Having plotted all the courses, proceed to fill in the interior work from the references and sketches shown on the right-hand page of the notebook.

In inking the map, use only colors that will wash. A diluted solution of bichromate of potash mixed with India ink will prevent spreading of the lines when touched with a wet tinting brush.

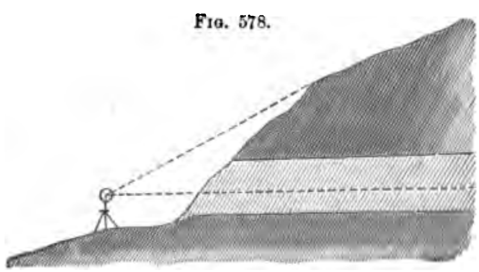
The map should show all the survey stations, stoppings of entrances, inclination of strata, and elevation of the stations above tide or other datum.

When different "levels" are to be represented, with their connecting shafts, etc., "isometrical projection" has been used, but "military or cavalier projection" is best.

LOCATING NEW LINES.

849. Second Object. To determine, on the surface of the ground, where to sink a shaft to meet a desired point in the underground workings.

FIG. 578.



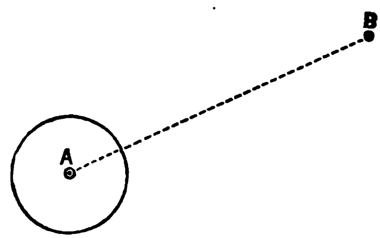
To do this, repeat on the surface of the ground the survey made under it—i. e., trace on it the courses and distances of the galleries, or their equivalents.

The chief difficulty is to get a starting point, and to determine the direction of the first line.

850. When the Mine is entered by an Adit (Fig. 578). Set the theodolite at the entrance, and get the direction of the adit and prolong it up the hill—i. e., in the same vertical plane. The third adjustment is here important.

If the line has to be prolonged by setting the instrument farther on, the second adjustment is important.

Fig. 579.



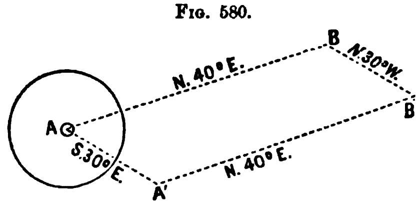
851. When the Mine is entered by a Shaft. Get the magnetic bearing of the first underground line, at the bottom of the shaft, with great care. Bring up the end of the line through the shaft by a plumb line, and set the compass over this point. Set out a line with the same bearing and length as the first underground line, and repeat the succeeding courses.

WHEN THE COMPASS CAN NOT BE SET OVER THE POINT, proceed thus:

1. Find, by trial, a spot, as B (Fig. 579), which is in the correct course, and measure off a distance equal to the length of the first underground course, and then proceed as before.

2. *Otherwise.* Set up anywhere, as at A′ (Fig. 580); take the bearing and distance of A from A′; run a line corresponding with the one underground, from A′ to B′. Repeat the course A′ A from B′ B; then A B is the desired line.

Fig. 580.



852. To dispense with the Magnetic Needle. *First Method.* — Let down two plumb lines on opposite sides of the shaft, so that their lower ends shall be very precisely in the underground line (see Art. 845).

Second Method.—Set, by repeated trials, two transits on opposite sides of the shaft, so that they shall at the same time point to one another, and each, also, to one of two points in the under-

ground line. They will then give the direction of the line above ground.

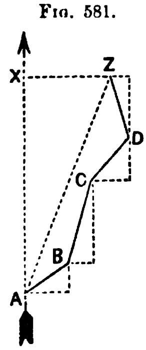
Third Method.—If the telescope of the transit be eccentric, as in Fig. 575, set the instrument on a platform over the mouth of the shaft, so that the line of collimation of the telescope shall be in the same vertical plane with two points in the underground line, on opposite sides of the shaft. When the instrument is so placed that, in turning the telescope, the intersection of the cross hairs strikes the two points in the underground line, the line of sight, when directed along the surface, will give the required line.

853. Having determined the first line, the courses of the underground survey may be repeated on the surface; or the bearing and length of a single line be calculated, which shall arrive at the desired point.

Let the zigzag line, A B, B C, C D, D Z (Fig. 581), be the courses surveyed underground, A being an adit, or at the bottom of a shaft, and Z the point to which it is desired to sink a shaft. It is required to find the direction and length of the straight line A Z.

When the compass is used, calculate the latitude and departure of each of the courses, A B, B C, etc. The algebraic sum of their latitudes will be equal to A X, and the algebraic sum of their departures will be equal to X Z. Then is tan. $Z A X = \frac{X Z}{X A}$; that is, the algebraic sum of the departures divided by the algebraic sum of the latitudes is equal to the tangent of the bearing. The length of the line A Z equals the square root of the sum of the squares of A X and X Z; or equals the latitude divided by the cosine of the bearing.

FIG. 581.



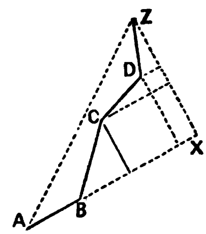
When the transit is used, instead of referring all the lines to the magnetic meridian, as in the preceding case, any line of the survey may now be taken as the meridian, as in traversing.

In Fig. 582 all the courses are referred to the first line of the survey. As before, a right-angled triangle will be formed.

Tan. $Z A X = \frac{X Z}{X A}$, and the length of $A Z = \sqrt{\overline{A X}^2 + \overline{X Z}^2}$; or $A X \div \cos. X A Z$.

Two or more lines may be substituted for the single line in the two preceding cases; the condition being that the algebraic sums of their latitudes and of their departures shall be equal to those of the underground survey.

FIG. 582.



854. Third Object. To direct the workings of a mine to any desired point.

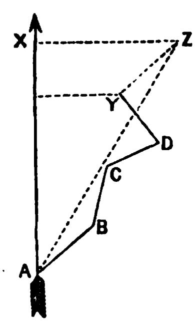
This is the converse of the second object. We repeat under the ground the courses run aboveground; or their equivalents, as in Art. 853.

In Fig. 583, let A B, B C, C D, D Y, be the present workings of a mine, and Z the shaft to which the workings are to be directed.

Find the latitude and departure of A Z. Then the difference between the algebraic sum of the latitudes of the underground courses already run and the latitude of A Z is the latitude of the required course; and the difference between the algebraic sum of the departures of the underground lines and the departure of A Z is the departure of the required course.

The length of Y Z equals the square root of the sum of the squares of its latitude and departure.

FIG. 583.



855. Problems. Most of the problems which arise in mining surveying can be solved by an application of the familiar principles of geometry and trigonometry:

1. Given the angle which a vein makes with the horizon and the place where it meets the surface, to find how deep a shaft at D will be required to strike the vein:

$$D C = A D \,.\, \tan. D A C$$

2. Given the depth of the shaft D C and the "dip" of the vein, to find where it crops out:

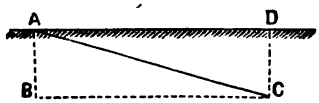
$$A D = D C \,.\, \cot. D A C$$

3. Given the depth of a shaft when the vein "crops out" and the "dip" of the vein, to find the distance from the bottom of the shaft to the vein:

$$B\,C = A\,B\,.\,\cot.\ A\,C\,B$$

If the ground makes an angle with the horizon, then the problems involve oblique-angled triangles instead of right-angled triangles, as in the preceding cases. Their solution, however, is quite as simple.

Fig. 584.



In the more difficult problems, the measurement of lines is required, one or both ends of which are inaccessible. (For a full investigation of this subject, see Part I, Chapter V.)

856. Mining Claim Boundaries. The methods of surveying the boundaries of a mining claim do not differ from the methods of plain surveying given in Part I. The law relating to the locating of mining claims differs in the different States and Territories, and it is necessary for the surveyor to know the law on this point in the locality where the survey is to be made.

In most cases he will be governed by the instructions issued by the Surveyor General of the State or Territory where the work is to be done.

CHAPTER XIX.

CITY SURVEYING.

357. General Plans. Few large cities were originally laid out according to any definite plan. Most of them extended and grew without any general design for the arrangement of streets. In certain cases this has necessitated a rearrangement in modern times of some of the streets, and the opening of straighter and wider ones to accommodate the increasing traffic. This is notably the case in Paris, where wide boulevards have been opened through the older parts of the city. Something in this line has been done in London, and more of it is planned for the near future. In most old cities unforeseen needs compel the opening of new streets by the right of eminent domain with the tearing down of houses and other buildings. Sometimes the object is to relieve congested lines of traffic, sometimes to obtain needed air space in the form of parks, and, again, to correct and abolish nuisances arising from overcrowding of buildings. In some cases the older parts of the city show the old no-system method, with its narrow streets winding and twisting in all directions, while the newer parts are laid out according to a well-defined plan. New York city is an example of this. It has been pointed out in the case of New York that in making the general plan of the city it was supposed the chief movement would be from river to river,* "but experience has not confirmed this theory, and the system of blocks is reversed from what it should be for up- and down-town travel."

It is the province of the municipal surveyor to intelligently anticipate future needs of all kinds when laying out a city or any extension thereof.

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* Report of Samuel McElroy to Town Survey Commissioners of Kings County, 1874.

In planning a system of streets for any town the design to be adopted will depend upon the contour of the ground and the natural features of the locality. If there be a water front, this will determine the direction of some of the main lines. If the site includes hills, the direction of the streets will be determined by the natural slope of the ground and the maximum grade adopted for the streets.

In hilly localities streets should be laid out, so far as it is possible to do so, in such a manner as to secure practicable grades for the streets at a minimum of expense for earthwork. Steep hillsides should therefore be ascended diagonally so as to gain distance and diminish the rate of grade. Many features, such as water courses, natural water power, canals, railroads, etc., will influence the direction of the streets in their vicinity.

At a seaside resort, where the object sought is a view of the sea, a rectangular system intersecting the shore at an angle of 45° has been tried with advantage.*

Usually the best plan for the streets of a city is that of two general systems running at right angles with each other, and a few diagonal streets to accommodate traffic moving obliquely to the main lines. The city of Washington is a good example of this plan † (see Fig. 585).

It should be borne in mind that the Federal Government assumes one half of the expense of maintenance of the streets in Washington. In many places a similar multiplicity of wide paved streets would occasion a nearly ruinous expense for pavements and maintenance.

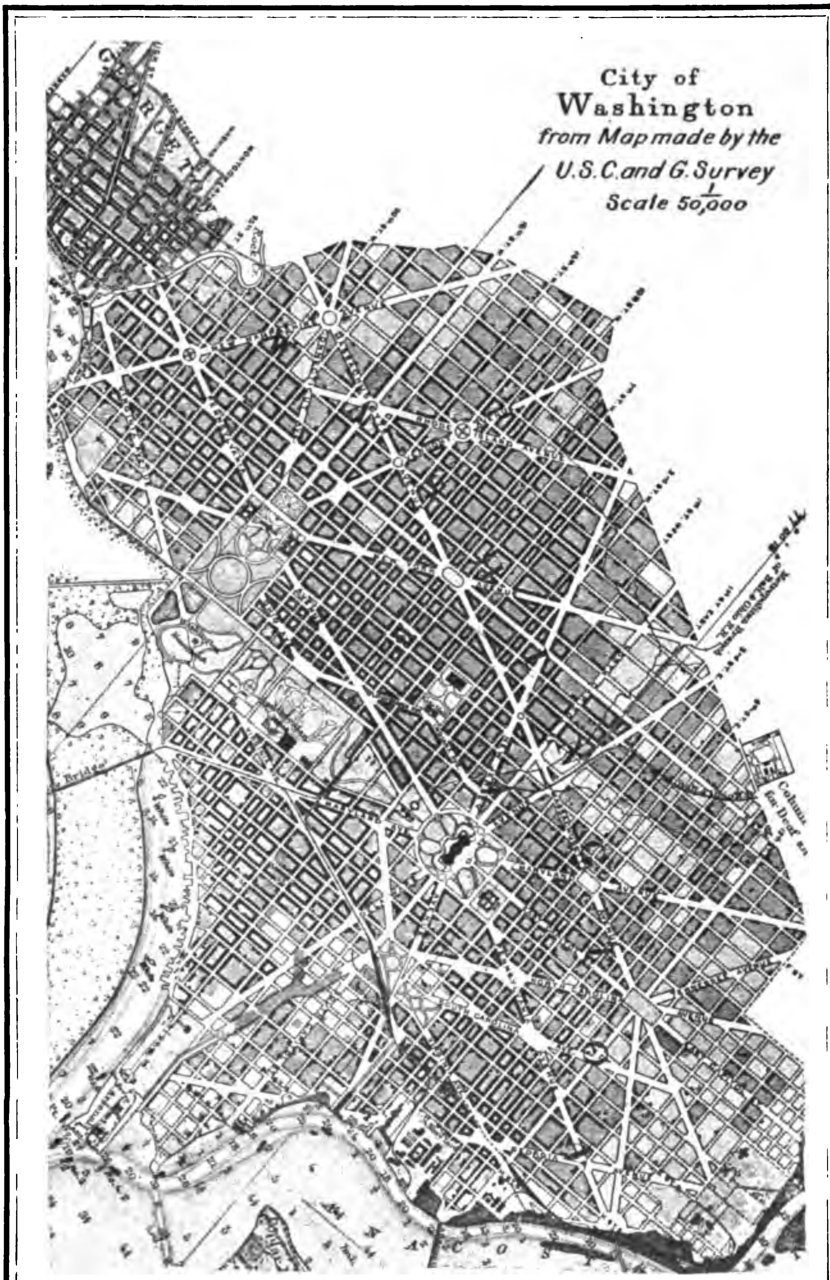
A system of streets at right angles to each other can not be extended unbroken over wide areas, excepting where the ground has only gentle slopes. Access to ferries, bridges, and country roads will often determine the direction of important streets, and in some cases will lead to a plan of radiating and polygonal streets, like a cobweb.‡ In most cases, however, a rectangular system may

* At Como, on the New Jersey coast.

† The plan of streets in Washington was originated by Pierre Charles L'Enfant, Major of Engineers during the American Revolution. Major L'Enfant laid out the city about 1790, and his plan has been but little changed.

‡ See "Engineering News," November 21, 1895, p. 344.

Fig. 585.


City of
Washington
from Map made by the
U.S.C. and G. Survey
Scale 1/50,000

be applied in sections adjusted to the natural features of the ground. There will, of course, be irregular lines where the sections join. Streets are generally laid out with straight or broken lines, although curved lines are occasionally employed.*

The problem most frequently presented to the surveyor is to lay out an addition to a town. In this case the new main streets must conform to the relief and natural features of the ground, and also connect properly with the present streets. Excepting where the hills are very steep the rectangular plan can usually be readily applied, and so arranged as to make the new streets continuations of the old ones. No plan will fit all cases; indeed, no two cases will probably be alike, unless where the ground is nearly level and unbroken by bodies of water.

858. Size of Blocks. There is a wide difference of opinion concerning the proper size of blocks and width of streets. In the village stage of a town, and especially in the residence portion, deep lots are desired for gardens, but for business streets it is not advantageous to make the lots very deep. Blocks may vary from 300 to 800 feet on a side, and be either square or rectangular. In one direction they will be the depth of two lots. If the lots are each 200 feet deep the block will be 400 feet in one direction. In the other direction they may vary from 300 to 800 feet. No uniform rule can be given.† If the blocks are very short, too much land will be taken for the streets, and the cost of street improvements will be a heavy burden. If the blocks are very long, it will be inconvenient to pass from one side of a block to the other.

In very large cities residences tend to become crowded together, and it is best to plan for lots of ample size, and at the same time not to lay out blocks so large that they will require to be subdivided, thus occasioning a worse crowding than if proper foresight had been observed in the original plan. A large portion of New York was laid out by John Randall, Jr., between 1808 and 1820, in blocks measuring about 200 feet by from 610 to 920

* See "Engineering News," November 28, 1895, p. 368, as to curved streets in Washington.

† See W. H. Dorsey in "Engineering News" for August 29, 1891, p. 192.

feet; and blocks of similar proportions have been largely used in Brooklyn, where, over a large portion, "200 feet is the usual width of blocks, and in those cases east of Flatbush Avenue where they are 255, 593, and 265 feet, no advantage pertains to this excess."*

859. Width of Streets. The heaviest traffic and most important business houses will usually be found on streets running in one direction, while the streets running at right angles to these will, as a rule, be of less importance. Under these conditions it is best to make those streets widest which are in the direction of the heaviest traffic, so far as this can be foreseen. Important business streets should be from 100 to 150 feet wide. Since street cars have come to play so important a part in street transportation wider streets are demanded. A double-track street railway will cover about 14 feet of the width of the carriage way, and will practically monopolize 18 feet of the width. Residence streets may be from 50 to 80 feet wide.

860. Sidewalk Widths. In some cities the width of the sidewalks is uniformly 0·2 of the entire width of the street; in others, city ordinances prescribe a width obtained by adding 2 feet to 0·2 of the width of the street, if the latter is a multiple of 5; or, if not, to the next lower number that is a multiple of 5—i. e., a 66-foot street would have 15-foot sidewalks.

The ultimate expense of good street pavements should always be kept in view, and to this end the narrowing of the carriage way is often permissible in residence streets and wide business streets. It may often be advisable to alter the second rule for sidewalk width, as given above, by making the additive number 5 feet instead of 2 feet. The width of sidewalk should be fixed by city ordinance, for the exact length of curb must be known in order that the street grades may be correctly established.

The lots should be of such depth that no excuse can be given for projecting signs, steps, show windows, etc., into the street, so that the sidewalks may be clear of incumbrances so far as possible.

* Report of Samuel McElroy, Superintendent of Survey of Kings County, 1874.

861. Alleys. There are conveniences attendant on the use of alleys running through the centers of the blocks parallel to the principal streets. These should be from 15 to 20 feet wide, so that two wagons may pass.

The use of alleys should be carefully guarded by proper municipal ordinance, otherwise they may become nuisances from the erection thereon of poor and small houses, sheds and barns, and from accumulations of refuse, etc. Alleys render the proper police protection of a city much more difficult, while they may facilitate protection from fires, unless they are allowed to be encumbered with buildings of an improper description.

862. Monuments and Reference Points. Permanent monuments and reference points for surveys increase in importance as the land rises in value. In cities, where land is valuable, all surveys should be based on permanent monuments, which can be readily referred to. Monuments should usually be set in advance of all improvements, and they should be carefully referenced and reset when disturbed by grading operations. If possible, they should be placed at all street intersections. For an original layout they may be at the intersections of the center lines of the streets, or on a line parallel to the center line, but these monuments will need replacing before the streets are improved. Where there is much traffic, and especially where, as is usual in cities, car tracks, sewers, gas and water pipes, occupy the center line of the street, it is more convenient for use to have the monuments outside of the carriage way, on the line of the curb, or a given distance inside of the curb, so as to come in the sidewalk. The monument may generally be placed on the prolongation of a street line, and three or five feet distant from a corner intersection, with minimum chance for disturbance, and in a position for convenient use.

Wooden posts, iron rods, gas pipe, and terra cotta have all been used for marking important reference points, but the best material for general use is stone. Cast-iron pipes or pieces of old rails also make excellent monuments, and are frequently less expensive than stone. A few of the principal monuments should extend below the frost line, or, better, should be supplemented with underground ref-

erence marks. Enough of these should be set to be available to test the position of all the other monuments on a line, in case there should be any question as to a monument having been moved. The remainder of the monuments may be stone blocks about three feet long, six to eight inches square at the top, and ten to twelve inches square at the bottom. The top and bottom should be perpendicular to the axis, and the station point should be marked on the top with a cross, or by a small hole in which to set the point of the sighting pole. Stone monuments, with the top cut square, and having one corner cut away, are used in great numbers in New Haven, Conn., the corner diagonally opposite the one truncated being placed to mark the street corner, in such manner that the monument is entirely in the street, and therefore does not encroach on private property.*

All surveys, whether of private lots or streets, should be referred to the monuments, and the positions of the latter should be indicated on all plots and maps. All levels should be referred to a common datum. In towns with water fronts the height of mean low water is a convenient datum. In any case the plane of reference should be taken below any points in the town. The mean level of the sea is the customary datum in extensive surveys, and in many European states this datum is in general use in the cities; it is also the plane to which triangulation is universally reduced.

Bench marks should be placed on permanent available points for all city work. Door sills, water tables of houses, or spikes driven into joints of walls, make convenient bench marks.†

Monuments and bench marks make it possible to execute work in any locality in the city, and to have all measurements, whether

* For a more extensive discussion of this subject, see "Transactions of the American Society of Civil Engineers," July, 1894; Broomall on "Marking of Street Lines, with Discussion." See also "The Final Results of the Triangulation of the New York State Survey, 1887, Appendix B," where the method of station recovery, adapted from that given by Prof. Marek, is especially worthy of note. See also as to the use of lime and stakes for marking points, by George C. Power, member of the American Society of Civil Engineers, in "Engineering News" for July 19, 1894, page 55.

† See discussion on Broomall's paper already referred to; also "Experiments on the Stability of Bench Marks," by G. W. Cooley, in "Transactions of the American Society of Civil Engineers," February, 1889.

of distances, directions, or levels, referred to permanent points whose precise positions are known.

863. Instruments. For city surveying much more accurate instruments should be used than would suffice for ordinary land surveying. The compass and chain should be discarded. All angles and directions should be determined with the transit, and, in general, all distances with the steel tape. For city triangulation a finely divided limb is necessary, its readings being to ten seconds or less, the usual methods of precise angular measurement being employed. Heliotropes are sometimes useful in city work. When they are used in connection with angle measurements of a triangulation, a mirror one inch in diameter will be of ample size, and the light may be subdued by means of gauze or crape screens, one or more as needed. Leveling instruments should have firm tripods, and the "levels of precision" used on geodetic work could often be employed to advantage in a city for running lines of benches, etc. A properly equipped municipal engineer should have at least one transit of superior workmanship, and one level of similar high grade, to be kept in reserve and not used for the many less precise operations he may be called upon to perform. It is important, also, that a standard tape be kept in reserve, with a certificate of its correctness from official sources—e. g., the Bureau of Weights and Measures at Washington. A field thermometer, spring scales, plumb bobs, marking pins, sighting rods, etc., are indispensable. The little known instrument, the measuring wheel or odometer, is often useful. By its aid a rodman can measure distances and also perform his ordinary duties for the leveler. Depending on the nature of the surface measured, wheel measurements can be made with a mean error of from 2·5 feet to 16 feet per mile. Leveling rods should be of a pattern that can be read from the instrument, though a target should generally be attached to the rod for use in driving stakes to grade. For this last-named purpose sights of considerable length must frequently be taken, and the target should therefore be of a pattern that will allow a white space to be bisected by the horizontal wire. A target that can be attached to a sighting pole is of great use in accurately grading a number of stakes, serv-

ing as a permanent reference mark to test the vertical as well as the horizontal alinement, and guarding against accidental disturbances of the instrument. Stadia lines should be added to the transit. It is well to bear in mind that the ultimate end of almost all stadia measurement is plotting on a small scale map, hence laborious numerical work is generally out of place; all the reductions should be made graphically, or by means of the slide rule.* This last-named instrument is invaluable to an engineer for the vast multitude of trial computations, interpolations, etc., that he is daily called on to make. The magnetic needle is used very infrequently, and may be entirely dispensed with for most of the work of a city surveyor. The staking-out of works of improvement, sewers, curbs, etc., ordinarily demands transits and levels of secondary precision, and as work of this nature is frequently hazardous to the instruments, those held in reserve should not be used for such purposes. The steel tape is in almost universal use for the accurate measurement of distances. Brass tapes were used for the careful surveys of the Back Bay district of Boston, and possess many merits.† Mr. Samuel McElroy devised an apparatus by which a No. 13 steel wire 600 feet long, stretched between end points, served as a medium upon which the measurement, with a 12½ foot wooden rod and clamps attached to the wire, was carried out. This apparatus was used with great success in laying out 2,000 miles of streets, covering over 36 square miles, in the suburbs of Brooklyn.‡

864. Subdivision of Blocks. When the blocks are rectangular the side lines of the lots will naturally be run perpendicular to the front line; but where the blocks are not rectangular, various methods have been used for dividing them. Often all the side lines of the lots are run obliquely to the front, and in some cases each line has a different angle with the street line. It is better to arrange the subdivision so that as many of the division lines as possible will

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* See Jordan's "Vermessungskunde," "Ausarbeitung der tachymetrischen Aufnahmen."

† See "Engineering News," January 12, 1889, page 32; also page 171.

‡ See "Report to Town Survey Commission," by Samuel McElroy, Superintendent, 1874.

be perpendicular to the street line, and all of the irregular lots be limited to the corners of the blocks. Where the two opposite street lines of a block are not parallel, the center line of the alley through the block, or the dividing line of the block where there is no alley, is run so as to bisect the angle formed by the two street lines. It will then be, at any point, equidistant from the street lines.

865. Marking Lot Corners. Lot corners are usually marked with stakes, which serve temporary ends only. Where buildings or fences may not be erected for a considerable time, agricultural drain tiles may be set with a post-hole auger, with a rough stone as a surface mark or finder. Where buildings are to be erected the stakes should be outside of the lines of the excavation, and on the prolongation of the neat lines of the building. The builder or architect should always be provided with a sketch showing the location of the stakes, and a similar sketch should be made in the surveyor's notebook, for a mistake or misunderstanding frequently leads to much trouble, and the correction may be a costly matter.

866. Resurveys and Lot Location from Deeds. The most vexatious work of the city surveyor is that of harmonizing discrepancies, correcting errors of former surveys, locating mistakes of inaccurate and careless surveyors who have preceded him, allowing for differences of standards of length, attempting to reproduce directions given by the magnetic needle under uncertain conditions and frequently at unknown dates, and, finally, discovering and, if possible, correcting clerical errors in deeds and descriptions of pieces of land. It is impossible to give any fixed rules for work of this description, where from the nature of the case the data are never the same. Careful study of the facts, good judgment and experience, combined with an earnest desire to do justice not only to one's own employer but to the innocent victims of the mistakes that may be discovered, will generally lead the surveyor to a solution of the problem before him. Frequently an explanation of the facts to all interested may lead to amicable adjustment. The supposed encroachment of neighbors is often the cause of much strife, and the surveyor should

always be sure of his facts, realizing that his testimony may be called for by the courts.

It is a hazardous venture for a surveyor to attempt to locate city property without first working in co-operation with those more experienced, and carefully studying the conditions peculiar to the city.*

867. Street Grades. As streets are fixed in horizontal alinement by reference to monuments, in like manner they must be fixed in profile by reference to well-established bench marks. Horizontally the street lines are supposed to be unchangeable, but vertically the line least subject to change is that of the curbstone.† The ideal street grade is one that fixes the curbstone on a straight line between intersecting streets, and which makes the opposite curbs everywhere of the same height. This arrangement is not always practicable, but in general, at street intersections, the grade of the crossing between curb lines should not exceed 3 per cent. Between the curb intersection and the house line the grade should *never* go beyond 8 per cent, and, unless the difficulties are extreme, 4 per cent should not be exceeded.‡ Usually, even in hilly cities, the grades of streets can be kept less than 5 per cent; if this is exceeded, the street pavements and sidewalks are apt to become dangerous in wet and icy weather. Vertical curves are often necessary to avoid abrupt transitions in grade. If m and n represent the rates per cent of two grades that are to be rounded off at their junction by means of a curved (strictly speaking a broken) line of p stations, of equal length, then $\frac{n-m}{2p}$ will represent the change in rate per cent at the first and last points of tangency, and $\frac{n-m}{p}$ the change at intermediate stations. For example, a street with descending grade of $-2{\cdot}64$ per cent $(=m)$ abruptly changes to a rising grade

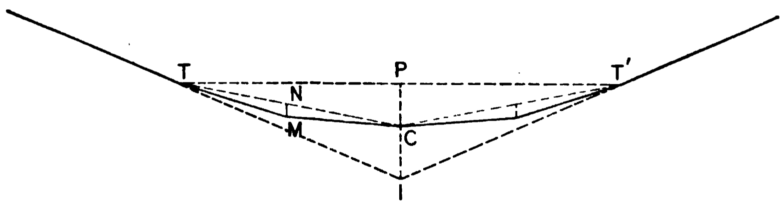
* See, in this connection, "School of Mines Quarterly," March and April, 1893, and "Engineering News" for June 8, 1893; also Duckham on "Retracing Old Lines, etc.," in "Engineering News" for July 11, 1895, page 26.

† In England and Canada called "kerbstone," and in Boston "edgestone."

‡ See report of Hering and Rosewater, Consulting Engineers, in "Third Annual Report of the Board of Public Works of Duluth."

of $+3$ per cent ($=n$). It is desired to round the grades by means of six stations, each 25 feet in length ($p=6$). $\frac{n-m}{p}=+0{\cdot}94$ per cent. $\frac{n-m}{2p}=+0{\cdot}47$ per cent, and the grades are $-2{\cdot}64$ per cent, $-2{\cdot}17$ per cent, $-1{\cdot}23$ per cent, $-0{\cdot}29$ per cent, $+0{\cdot}65$ per cent, $+1{\cdot}59$ per cent, $+2{\cdot}53$ per cent, and $+3$ per cent. If the height at the first tangent point is 189·51, the heights of the points

FIG. 586.



at the ends of the 25-foot stations are 188·97, 188·66, 188·59, 188·75, 189·15, and 189·78. Often it is necessary in the field to stake out a curve without knowledge of the rate per cent of the intersecting grades. This is best accomplished by continuous bisections, as shown in the accompanying diagram (Fig. 586).

Measuring from I, the intersection of the grades, make $\mathrm{I\,T}=\mathrm{I\,T'}$, any convenient distance. Target from T to T′, with instrument at T, and place a grade stake at C, making $\mathrm{P\,C}=\frac{1}{2}\,\mathrm{P\,I}$. Then target from T to C, and at N, the center point of T C, place a stake with its top distant from T C by $\mathrm{N\,M}=\frac{\mathrm{P\,C}}{4}$. A stake may, if necessary, be placed midway between T and M, using an offset of $\frac{\mathrm{N\,M}}{4}$.

The instrument may then be set up at T′ (and C, if necessary), and the staking out completed. In general, the distances will be sufficiently close, whether measured horizontally or on the grade, and the offsets may be measured vertically.

The grade at the building line, or street line, is in general derived from that of the curb by giving a definite slope to the walk sometimes one half inch per foot. With well-laid walks 2 per cent to 3 per cent is sufficient; whatever rate is taken is usually fixed by city ordinance. To fix the height at a corner, the rule given in the

Duluth report, above cited, may be quoted: "The elevations of the curbs at the two opposite points should be added to the rise of the two walks from their respective curbs to the corner, . . . the average of the two heights thus obtained will give the desired elevation." From the corner elevation thus obtained the grade at the building line should gradually approach that derived in the usual manner, from the curb opposite, and reach this point of normal pitch of walk at a point fifty to one hundred feet distant from the corner.

Grades can best be indicated by figures on plans, with accompanying profiles, but may be given for the purpose of legal enactment after the following model:

Myrtle Avenue Grade.—Beginning at the west curb line of Lexington Avenue, at the established height of 181·45 feet; thence rising at the rate of 0·5 to 100 for a distance of 726·26 feet, to the west curb line of Robin Street, to the height of 185·08 feet; thence rising at the rate of 0·946 to 100 for a distance of 1,058·07 feet, to the east curb line of Lake Avenue, to the established height of 195·09 feet.

868. Cross Section of Street. The cross section of the carriage way between the curbs varies with the width and the relative elevations of the curbs. The curb height may be made six inches, as a general rule. The center of the street should be from $\frac{1}{60}$ to $\frac{1}{100}$ the width of the carriage way above the gutters; $\frac{1}{80}$ is a ratio that gives good results in general; with this the crown, in inches, will be 15 per cent of the width in feet. There is no apparent advantage in using a changeable ratio of crown to width dependent on the grade of the street, though this practice has been recommended.* There are practical difficulties in paving a street with variable crown, though this is sometimes necessary where the width varies, or where one curb is very much higher than the opposite one. In this case a shoulder may be paved from the curb to the gutter line, the latter being kept two feet or more out from the curb for this purpose, the attempt being made to have the opposite gutters differ not more in height than 3 per cent of the width between them.

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* See report of A. Rosewater, City Engineer of Omaha, December 31, 1894.

The best shape of cross section of a city street between curbs is a compromise between "two inclined planes meeting in the center of the road, and having their angle slightly rounded by a connecting curve"* and the arc of a circle,† adopted very frequently. Such an intermediate shape is practically that recommended by General Gillmore,‡ and agrees with that recommended by Codrington.* To obtain this form of cross section, imagine a level line drawn touching the curve of cross section at its middle point, the offset from this line at points 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 (the center of the carriage way being 0, and its semiwidth being unity) are respectively 0, $\frac{5}{32}$, $\frac{12}{32}$, $\frac{21}{32}$, and $\frac{32}{32}$ of the assumed crown, while the less desirable arc of a circle will be obtained by using the offsets 0, $\frac{2}{32}$, $\frac{8}{32}$, $\frac{18}{32}$, and $\frac{32}{32}$. Observing that the differences between the first-mentioned ordinates are given by the numbers 5, 7, 9, and 11, and between the second by the numbers 2, 6, 10, and 14, the difference of form becomes apparent, and the offsets are easily memorized. Where the cross section is a circular arc the tangent to the curve at the gutters will have a rate per cent equal to 400 $\frac{c}{w}$, where $\frac{c}{w}$ is the ratio of crown to width, while the tangent will be level in the center of the street. With the improved cross section recommended, the tangent to the curve at the gutters will have a rate per cent of 300 $\frac{c}{w}$, while at the center the tangent will have a per-cent rate of 100 $\frac{c}{w}$ theoretically, although in practice the center tangent will be level, owing to the flattening due to rolling or pounding the pavement. Where one curb is higher than the opposite one it will often be best to reduce the ratio $\frac{c}{w}$ to $\frac{1}{100}$ or less, and to use the circular-arc form of cross section, in order to prevent the formation of a large flat area on the side of the high curb. Where a street railway

* William M. Gillespie, "Road-making Manual," sixth edition, 1853, page 50.

† With a crown of $\frac{1}{80}$ of the width, the arc of a circle will be that described with a radius of ten times the width between curbs.

‡ "A Practical Treatise on Roads, Streets, and Pavements," fifth edition, 1885, p. 70.

* See article "Roads," in "Encyclopædia Britannica," ninth edition.

is in the center of the street, the usual form of cross section may be modified by truncating the width occupied by the tracks, placing all the rails on the same level.

869. Subsurface Lines. The problem of arranging the locations of subsurface conduits in a street is a serious one. The number of systems is increasing. Formerly, sewers, gas and water pipes were all that must be provided for; but now there are, in addition, steam pipes, pneumatic pipes, electric wire and cable conduits; and the number of underground systems is increasing yearly. Where the first pipes have been put in without regard to those which are to follow, the difficulty of placing the latter is greatly increased. Some general plan should be adopted, and the position of each system should be decided upon. Under ordinary circumstances, a good plan is to place the sewer, which must be laid at the greatest depth, in the center of the street, and the other systems on either side, care being taken, however, to allow of the construction of receiving basins. Where the only additional systems are gas and water pipes, one may be placed on one side of the street and the other on the opposite side. When sewer and water pipes are under the carriage way, house connections should be brought to the curbs in front of every house and vacant lot before the street is paved.

Where possible, all underground work should be carefully located and plotted, with references to the depths and grades, by profiles or figures, where, as in the case of sewers, these are essential.

In Paris the sewers are made large enough to contain the other systems of pipes. This arrangement enables additions and repairs to be made to the pipes without disturbing the pavements. The Paris sewers were, however, not built primarily for house drainage, and are somewhat defective for this purpose. Electric wires, if placed in the sewers with the gas pipes, might be a cause of dangerous explosions.*

870. Surveys for Sewers and Drains. Of all the underground conduit systems, the one requiring the most careful consideration in

* See "Special Consular Report on Streets and Highways in Foreign Countries," Washington, 1891, as to the arrangement of pipes in many foreign cities.

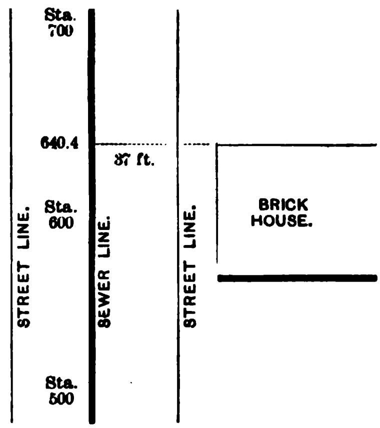
determining its arrangement is that of the sewers. Gas and water being under pressure, their pipes may run in almost any desired direction. But the flow of sewage, as a rule, depends upon gravity, and the pipes must be carefully located for their efficient operation. If a topographical map of a city exists, this will afford the readiest means for determining the natural lines of outflow. If no such map has been made, the heights at the street intersections, and at any intermediate points where there is any considerable change of grade, must be determined and noted on a plan of the city. From this map the best lines for the sewers are chosen. The position of all systems of pipes already on the ground should be determined, so as to avoid them, if possible, in locating the sewer trenches. In planning the lines, it is best to work from the outlet toward the upper ends of the branches. The line of the sewer should be located with a transit, and all measurements should be made with a steel tape. If the stakes were set on the center line of the sewer trench they could not be preserved during construction; they should therefore be placed on one side of the center line, at such a distance as to bring them one foot from the side of the sewer trench. In constructing the sewer, a space next to the trench, on the side where the stakes are, is kept clear for handling material used in construction, so that the stakes are always accessible.* Large spikes (60-penny) will be found more convenient for staking out sewer work than the wooden stakes usually employed for that purpose. The spikes can be driven into hard gravel or macadam roads, and they stand hard usage better. If possible, the offset line should be taken uniformly on the same side of the trench. All recorded measurements should be made to the center, and not to the offset line. On the preliminary location stakes are placed one hundred feet apart, and before construction begins others are interpolated twenty-five feet apart. The stakes are driven until the tops are even with the surface of the ground, and where the streets are paved a stone is taken up, the stake driven, and the stone replaced. When considerable time may elapse between the location and con-

* See "Municipal Engineering," November, 1896, p. 313. Also Rawlinson's "Suggestions," etc., London, 1878.

struction, reference points should be noted at frequent intervals to aid in finding the stakes.

The reference points may be fixed by either of the methods for determining the position of a point (Arts. 2 to 9), but the first and second methods are most used. The second method is most easily applied in sewer work, as follows: Note the point at which the prolonged line of the side wall of a building, or some other permanent range, meets the center line of the sewer, and also the distance from the center line to the structure. In Fig. 587 the prolongation of the side wall of a brick house meets the sewer line at station 640·4, and the distance to the corner of the house is 37 feet. To find any stake in the sewer line in the vicinity of the brick house, measure out from the corner of the house, and in a prolongation of its side wall, 37 feet. This gives a point on the line of the sewer 40·4 feet from station 600. Then by direct measurement along the center line any station in that locality may be found. By means of two such reference points the direction of the line is also determined.

FIG. 587.



871. Transit Notes. The transit notes should give a sufficient number of reference points on each block to locate the line of the sewer, and especially any change of direction. The intersections of both lines of all crossing streets should be noted; also all streams, railroad crossings, and whatever would in any way affect the building of the sewer.

872. Level Notes. The levels should be taken on the center line of the sewer, and carefully checked at the bench marks. For construction, levels should be taken at every twenty-five feet, and

CONSTRUCTION NOTES.

Greene Street.

Station.		Surface.	Grade.	Cut.	Y's. North.	Y's. South.
			·32 per 100. 8″ pipe.		North.	South.
0		14·9	6·12	8·8	0	Manhole.
	25	14·7	6·20	8·5		
Aug. 28, Sta.	40					41·5
	50	14·9	6·28	8·6	43·5	
	75	14·9	6·36	8·5	78·0	
						82·5
1		15·1	6·44	8·7	109·8	
						117·8
	25	15·3	6·52	8·8		
						143·7
	50	15·4	6·60	8·8	145·7	
						156·0
	75	15·4	6·68	8·7	157·4	Lamphole.
					183·3	
2		15·3	6·76	8·6		185·3
						218·7
Aug. 29.	25	15·4	6·84	8·6	220·7	
						230·7
	50	15·5	6·92	8·6		
					262·3	
	75	15·6	7·00	8·6		264·3
3		15·6	7·08	8·5	300·0	
						302·0
Aug. 30.	25	15·6	7·16	8·4		323·7
					335·2	Manhole.
	50	15·5	7·24	8·3	349·3	
						365·7
	75	15·5	7·32	8·2		
						392·0
4		15·6	7·40	8·2	404·0	
	25	15·6	7·48	8·1		426·0
						454·3
Sept. 1.	50	15·7	7·56	8·1	456·3	
						470·0
	75	15·6	7·64	8·0		484·0
					486·0	
5		15·4	7·72	7·7		506·0
Sept. 2.					512·0	Flush tank.
B. M. Arrow of hydrant cor. Jackson Street.						17·65
B. M. Top stone hitching post front of No. 224.						18·37

grade stakes may be given as frequently as that at the bottom of the trench, where it is practicable to do so. Where there is considerable side slope in the streets the heights at the street or curb lines should be noted, and also that of the cellar floors, on the lower side

Jackson Street to Brown Street.

Sta. 0 = Sta. 1268·7 Jackson Street.
Sta. 31·9 = W. line of Jackson Street.
Spikes 12 feet south of north curb.
Sewer 15 feet south of north curb.

Began work August 28.

6" water at Sta. 0; ran out of water Sta. 125.

Sta. 211·7 W. line of No. 235.

At Sta. 250, 25 feet B. M., sheeting left in to protect water service.

Allowed to Sta. 300 in estimate of September 1.

Sta. 300 to 512·0. Estimate of October 1.

especially. The level of all streams crossed should be taken, and also the height of the ground water, from neighboring wells. Borings are often needed to enable the nature of the digging to be judged from the profile.

873. Maps and Profiles. From the transit notes the lines of the sewers are laid down on the map and the size is indicated, with the positions of the manholes, lampholes, flush tanks, and other accessories. From the transit and level notes the profiles of the streets are made, showing the surface of the streets, the grade of the sewer, intersecting water courses, etc. Scales of 1 inch to 40 feet horizontal and 1 inch to 6 feet vertical are convenient or 80 horizontal and 12 vertical may be used. Uniformity is desirable, and engraved profile paper should be used.

874. Construction Book. From the transit notes, level notes, maps, and profiles the construction book is prepared. One form is as follows:

LEFT-HAND PAGE. RIGHT-HAND PAGE.

STATION.	SURFACE.	GRADE.	CUT.	CONSTRUCTION NOTES.	
				Ys	
				N.	S.

In the first column is given the station, in the second the height of the surface of the street, in the third the height of the sewer, and in the fourth the depth of the trench.

The right-hand page is filled in as the construction proceeds. Everything should be noted which is needed to give full information for the final record, such as junctions of laterals, house connections, peculiarities of material, rock, quicksand, etc.

875. Record Books. When the work is completed a final record should be made and preserved with the transit, level, and construction books. The records should be so arranged, and be so full and explicit, that all needed information regarding the sewers can be readily obtained. The form on pages 446 and 447 for the final record has given satisfaction.

The street name is given at the top of the page. On the left-hand page, the first column gives the station; the second, the height of the surface of the ground above datum; the third, the height of the sewer; the fourth is for notes concerning the appurtenances,

branches, manholes, etc. On the right hand-page a plan is drawn showing the sewer line, reference points, manholes, flush tanks, receiving basins, etc.

Another method of preparing a final record is to make a set of maps, each showing a small section of the town to a large scale (one inch to forty feet), and giving the street lines, pipe lines, branches, and all accessories. Necessary details can be written on the maps to give all needed information concerning the sewers.*

Other Pipe Systems.—The surveying for other systems of pipes—gas pipes, water pipes, etc.—is similar in method to that given for sewers. In some cases very little instrumental work will be needed. The trench may be placed a certain distance from the curb line, and accurate levels are rarely necessary. The records, showing the precise locations of the pipes and of all appurtenances, should be carefully made, and so arranged that all needed information can be readily obtained from them.

Where it is possible, a system of pipes—as, for example, water pipes—should be placed a uniform distance from the curb. Then, if the valves on the pipes are placed in the range of the bounding lines of the streets, they can always be readily and quickly found, even when covered with snow, without referring to notes or descriptions.

376. Surveys for other Kinds of Municipal Engineering Works. Where an old street is to be repaved, a profile should be made showing the heights of both curbs; then, after careful study of the profile, the old curbs should be tested by the process of "targeting," as described in Chapter XVI, and all breaks in the grade should be referred to some near bench mark, which can be chosen at random for this purpose. Vertical curves should be introduced where necessary, the attempt being made to improve the horizontal and vertical alinement of the new curbs as much as possible without detriment to steps, area ways, etc. Frequently it may be advisable to

* See Rawlinson's "Suggestions as to the Preparation of District Maps," etc., London, 1878. Also a series of articles on "Municipal Engineering," in "Engineering News" for 1886. Also specimen sewer map in "City Yearbook of New Haven, Conn.," 1873, p. 128.

relay most of the sidewalk when the new curb is placed. The preliminary work referred to should all be completed before paving material is piled on the sidewalks. In staking out work, that method should be employed that will enable the mechanics to do their work as easily as possible and with the tools at their command—cord, plumb lines, carpenter's levels, and pocket rules generally. Thus, in giving curb lines it is best to drive stakes of

Jackson St.

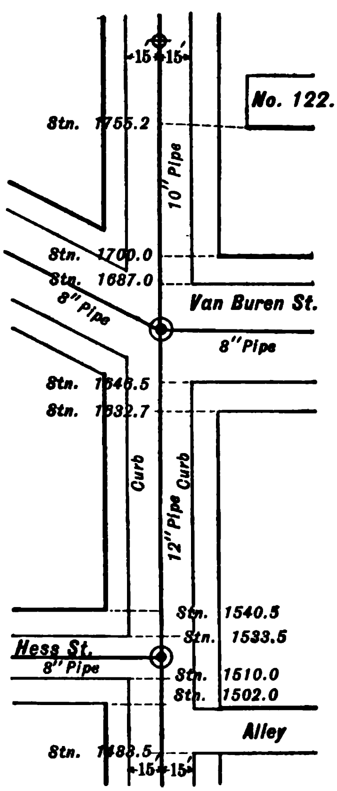
Station		Surface	Grade	Notes	
				1790.2	Lamphole
	75	16.2	6.70	1785.4	Y West
				1759.2	Y East
	50	16.2	6.64	1735.2	Y West
				1733.2	Y East
	25	16.0	6.58	1727.2	Y East
17		16.1	6.52		
	75	15.9	6.46		
				1666.8	Manhole
	50	15.8	6.40		
	25	15.6	6.34	1630.9	Y West
				1612.8	Y East
16		15.5	6.28	1606.8	Y West
				1584.8	Y West
	75	15.5	6.22		
	50	15.4	6.16	1552.6	Y East
	25	15.3	6.10	1522.0	Manhole
15		15.2	5.85	1509.7	Y East

hard wood, three and a half feet long, on the lines of the curbs, with their tops exactly at grade. These stakes should be about sixty feet apart, or nearer if necessary, and should be tacked for line. If the street is wide, or street railways exist, stakes will be needed in the center also, and, in general, near manhole tops, etc.,

which must be set to grade in advance of the pavement. In staking out retaining walls, pits for foundations, etc., stakes should be set at a certain height above the fundamental lines of the plan, and so arranged that cords may be drawn from one to another, giving the main outlines of the proposed structure.

It will be advisable to reference stakes where, as is generally the case, there is danger of loss or disturbance from caving banks, the

Monroe St. to Marshall St.



piling of earth or building materials, etc. It is much easier and more accurate for the surveyor to " plumb down " from considerable heights, with his transit, than it is for the mechanic to do so with a plumb line, especially when the points to be transferred vertically are difficult of access. Street cross sections are laid out by means

of a cord drawn taut from curb to curb, and from this measuring to the tops of pegs or iron rods * driven to the requisite height.

A template can often be used in streets of uniform width of carriage way. This template may either span the entire carriage way, or reach from the curb to the street-railway track, or to a timber laid to correct grade in the center of the street.

877. Street Railways. The usual practice of surveying and staking out railroads must be modified by the necessity of following the center lines of streets, and using curves of very short radius. It is best, in the case of a double-track railroad, to put all four rails on the same level, if the opposite curbs are of the same height; if such is not the case, the rails of each track can be made level, and one track can be raised to some extent above the other. By this means objectionable and dangerous side slopes in the street pavement, between the curb and the rail, may be avoided or lessened. To diminish the evils of short-radius curves, spirals are generally introduced, which render the transition from tangent to curve easier and more agreeable.†

The general use of heavy girder rails for street railways demands a careful location and referencing of points, since the rails are usually bent to proper form for the curves, and provided with frogs, turnouts, etc., at the manufactory.

878. Surveying and Mapping a City. The need of carefully made maps of our cities, showing the various pieces of property, buildings, curb lines, sewers, water pipes, etc., becomes more evident as age adds to the complexity of the original plan. Countless transfers of property, the erection and demolition of buildings, opening of new streets, and various other changes, demand a most careful survey of the older parts of a city; while, to anticipate future growth, in the suburbs, an accurate but less detailed and precise

* See "Special Consular Reports on Streets and Highways in Foreign Countries," Washington, 1891, p. 574.

† See, for example, Würtele on "Spirals and their Use on Railroads," "Transactions of the American Society of Civil Engineers," March, 1894; also valuable articles in "Engineering News" for July 23, October 15, and October 29, 1896, and for February 4, 1897.

topographical survey is needed. The fundamental features of a survey of an extended area have been described as follows: "The foundation of a properly conducted survey of a large territory consists in the determination of the astronomical places of a few principal points, a many-membered triangulation and a net of levelings of precision. For detail surveys the results of the triangulation should be expressed in rectangular co-ordinates, and the results of the leveling in heights referred to a common horizon, which should coincide as nearly as possible with the mean level of the sea." *

This statement will apply to a survey of a city and its suburbs, but in this case latitudes and longitudes are matters of very subordinate interest, and have no practical importance. It is desirable that there should be a carefully determined azimuth, while rectangular co-ordinates are essential. The triangulation of the Federal surveys is so far advanced that many of our larger cities have in their neighborhood triangle sides which may be used as a base of operations. It is best, in general, to regard one favorably located side as the base for the municipal survey, using other portions of the Federal triangulation, where convenient, to check the city work but not to serve as its basis. Of course the Federal work should not be used unless the identity of the points and the stability of the marks are beyond question, and unless assurance as to the exactness of length and direction of the initial line can be had. It is best to reckon azimuths in conformity with the usage of our Federal (and many foreign) surveys, namely, to take the south as the starting point of directions, and to reckon around by west through the whole 360°. This will make the meridian the axis of abscissas (x, + to the south, − to the north), and the perpendicular will be the axis of ordinates (y, + to the west, − to the east).

Strictly speaking, the azimuths are true only in the meridian of the origin, no allowance for inclination of meridians being made or desirable. If for any purpose it is necessary to determine the inclination of the meridians at any point east or west of the origin,

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* Resolution adopted at the sixth convention of the Society of Surveyors of Germany, August, 1877.

the computation is a simple one.* In general, however, the exact direction of the cardinal points is of little interest, and is useless for the purposes of the survey, which could be carried to completion just as well with entirely arbitrary directions for the co-ordinate axes. Indeed, where the general directions of a rectangular system of streets is askew with the meridian, co-ordinate axes parallel with the streets would facilitate the map work, and the meridians could be added to the maps if desired.

The origin may be taken preferably at some triangulation point (or imaginary point, if no convenient triangulation point exists) lying entirely southeast or northwest of the area to be surveyed. The co-ordinates of any point will then always have + and − signs, showing at a glance which is the ordinate and which the abscissa; moreover, there are other and more important practical reasons why it is best to avoid variable signs for either abscissa or ordinate.

For the purposes of a city survey the co-ordinates may be regarded as *plane* without appreciable error.†

If a base line can not be obtained from the Federal triangulation one must be measured, and a check base should be measured in any event, the methods developed in Chapter XI being used. In general, a steel tape should be used, and the base should be measured under such conditions that the temperature of the tape can be found with precision; hence a calm, cloudy, or rainy day, or a night when the temperature is nearly constant, would be suitable times.‡

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* The necessary formula is given in the "Final Results of the Triangulation of the New York State Survey," 1887, p. 174.

† For a detailed discussion concerning the effect of the earth's curvature on the co-ordinates, etc., see Jordan's "Vermessungskunde," vol. ii, chapter vi; also Börsch's "Geodätischer Co-ordinaten," Cassel, 1885; or a paper by Horace Andrews in the "Final Results of the New York State Survey." The matter of rectangular co-ordinates is discussed at great length in the "Zeitschrift für Vermessungswesen," 1872 to 1896.

‡ See, as to steel-tape measurements of precision, the "United States Coast and Geodetic Survey Report" for 1893, Appendix V, also for 1894, Appendix VI, and a paper by R. S. Woodward in the "Transactions of the American Society of Civil Engineers" for October, 1893; also report of Captain J. H. Willard, 1893, Appendix V of the "Annual Report of the Chief of the Corps of Engineers"; also Johnson's "Surveying."

879. Primary Triangulation. The triangulation should be so developed that the entire area of the survey will be covered by a network containing a few large triangles. If the method of measuring angles in "sets" or series is adopted, to attain a high degree of precision, from ten to twelve of these, in both positions of the telescope, should be taken, each point being sighted four times in each set. Signals should be adapted to the length of sight. Flat targets, or vanes, are capable of more precise bisection than poles. Heliotropes are sometimes used, with screens of gauze or black crape to subdue the light—an essential precaution for accuracy of pointing on short sights not exceeding eight or ten miles. In general, the primary triangulation should possess not more than eight or ten equations of condition, one or two of these being side equations, and the adjustment should be by the method of least squares. In precise work the mean error of angle measurement varies from 0·5″ to 0·75″. Adjustment should not change the value of any angle by more than 2″ from that observed, and the mean error of co-ordinates should be about two centimetres.

880. Secondary Triangulation. From the points of the primary triangulation those of the secondary; church spires, pinnacles of buildings, and points selected on roofs and marked with small poles, generally unoccupied points, are to be located by "forward cuts," as many cuts being taken to each point as possible. The measurements should be in sets of six to eight observations each, in direct and reverse. Objects so close as to have large angular diameter should have the sights equally divided between the right-hand and left-hand edges. The measurement of angles of the primary and secondary triangulation demands a good eight-inch or ten-inch theodolite, reading with micrometers, if such can be obtained. Angles of secondary triangulation should not be changed more than 5″ by adjustment, and mean errors of co-ordinates should be about fifteen millimetres. Adjustment may be effected either by the method of least squares or graphically.*

881. Tertiary Triangulation. The points of the tertiary triangulation are to be on the ground for the most part, and in general are

* See "Final Report of the New York State Survey," pp. 31–35.

to be so chosen that the traverse lines can be joined thereto. A smaller instrument may be used for the triangulation of the third order, with four to six full sets on all observed objects. Triangulation points of this order are located partly by forward cuts from the primary points, and chiefly by "backward cuts" from the tertiary points themselves. Sometimes four orders of triangulation are recognized, there being an order interpolated between the first and second, as herein described, with occupied stations for the better determination of the secondary points.

The unoccupied points of the secondary system are often connected with the traverse work by being "brought down"—that is to say, by means of a short, carefully measured base, from one end of which, at least, some other triangulation point can be seen. The problem of "backward cuts," above referred to, is an amplification of the three-point problem and is very useful in locating tertiary points.* The mean error of co-ordinates of the tertiary points may be expected to be about eight to ten millimetres.

It is customary in accurate city surveys to mark all occupied triangulation points with care, using stone or iron monuments or other durable marks. The number of triangulation points will, of course, depend upon the extent of the survey and the refinement of the traverse work. Many German cities have been surveyed with such minuteness and care that it would be possible to reproduce the entire ground plan of the city and to relocate all essential features if the triangulation points, with some of the main traverse points, remained intact.

In the survey of Berlin about twenty-four square miles were covered, with 563 triangulation points, over one third of which were located by backward cuts.

Over 100,000 separate pointings were required in this very detailed triangulation, which was carried to such a degree of completeness that very little proper traverse work of the first order was needed. In general, it will be necessary to have from three to seven traverse points between triangulation points.

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* See Jordan's "Vermessungskunde," vol. i, 1877, p. 357, etc.; also "Report of the United States Coast and Geodetic Survey," 1864, Appendix XIII.

882. Traverse Lines. Traverse lines of the first order should be in nearly straight lines between triangulation points; the sides should be of nearly the same length; they should follow principal streets, and should not, as a rule, serve for detail work. Second-order traverse lines join on triangulation points, on traverse points of the first order, and also, to avoid an unfavorable layout of the principal traverse, upon intermediate points in the sides of the latter, carefully lined in and measured to for this purpose.

The secondary traverse lines follow by-streets, and serve, as a rule, for detail surveys. Third-order traverse lines are bound to those of the first, second, and third orders, and serve exclusively for detail work; they are therefore run near house lines (within three or four feet) and within blocks, courts, etc.

The average bowing out from a straight line should not exceed 5° in the main traverse, and about 17°, as a maximum, in the subordinate.

Traverse points should be marked or referenced with stone posts, iron pipe, spikes, etc., by witness marks, bolts, or tacks in the walls of neighboring houses, and more especially by prolonging the traverse lines backward to a neighboring wall or house front, where a permanent mark should be made and measured to. Where possible, it is advisable to have a traverse line limit the work of each sheet of the city plans. The traverse points and lines should be chosen so as to be out of the way of the traffic, as far as this is possible.

883. Measurement of Angles. Extreme care in centering both transit and sighting object will be necessary. In traverse work of the first and second order, angles should be measured in three different positions of the circle, and each time in direct and reversed positions of the telescope. In the third order a double measurement, in direct and reverse, will be sufficient. Always begin with zero on the last occupied point, or at starting, on a triangulation point.

884. Linear Measurement. Always measure each line twice, in opposite directions. Measurements are best made on the ground, without plumbing, by leveling to changes of grade and subsequent

reduction to the horizontal. Sometimes measurements are made with steel tapes, with always a careful record of the temperature of the tape from time to time, and with a spring scale for tension. Sometimes the measurements are with wooden rods five metres long; and in most German cities these are regarded as better and more precise than steel tapes, chiefly on account of their small change with temperature. Both tapes and rods should be carefully compared with a standard, the rods frequently, and the tapes as often as broken and mended. Rods are subject to a slow change in length from atmospheric influences. The fragility of steel tapes is a source of constant and great annoyance. Breakages are especially liable to occur where there is heavy traffic. With steel tapes the correction for temperature can be made under the assumption of a change of $\frac{1}{10}$ of 1 per cent in length for a change of 15° F. in temperature.

Corrections for inclination can be made as follows:

Let b = base, h = hypotenuse, and p = the perpendicular of a right-angled triangle, then for ordinary slopes it may be assumed that $b = h - C_1 \ldots.$ [1.]

where $C_1 = \frac{p^2}{2\,h}$; or, if a closer approximation is desired,

$$b = h - C_1 - C_2 \ldots. \qquad [2.]$$

where $C_2 = \frac{C_1^{\,2}}{2\,h}$, or, for a still closer approximation,

$$b = h - C_1 - C_2 - C_3 \ldots. \qquad [3.]$$

where $C_3 = \frac{C_1\,C_2}{h}$

For example, taking an extreme case, a distance of 100 feet measured on a slope where the perpendicular is 20 feet, would have $C_1 = 2$ feet, $C_2 = 0{\cdot}02$ feet, and $C_3 = 0{\cdot}0004$ feet, the resulting horizontal distance 97·9796 feet being correct to $\frac{1}{10{,}000}$ of a foot. The slide rule is of great service in computing the corrections for temperature and inclination.

The greatest permissible difference between the two measurements of a traverse line of the first or second order should not exceed $0{\cdot}05\sqrt{d}$ in feet, where d is expressed in units of 100 feet—i. e., for $d = 400$ feet, the greatest difference between the two measurements

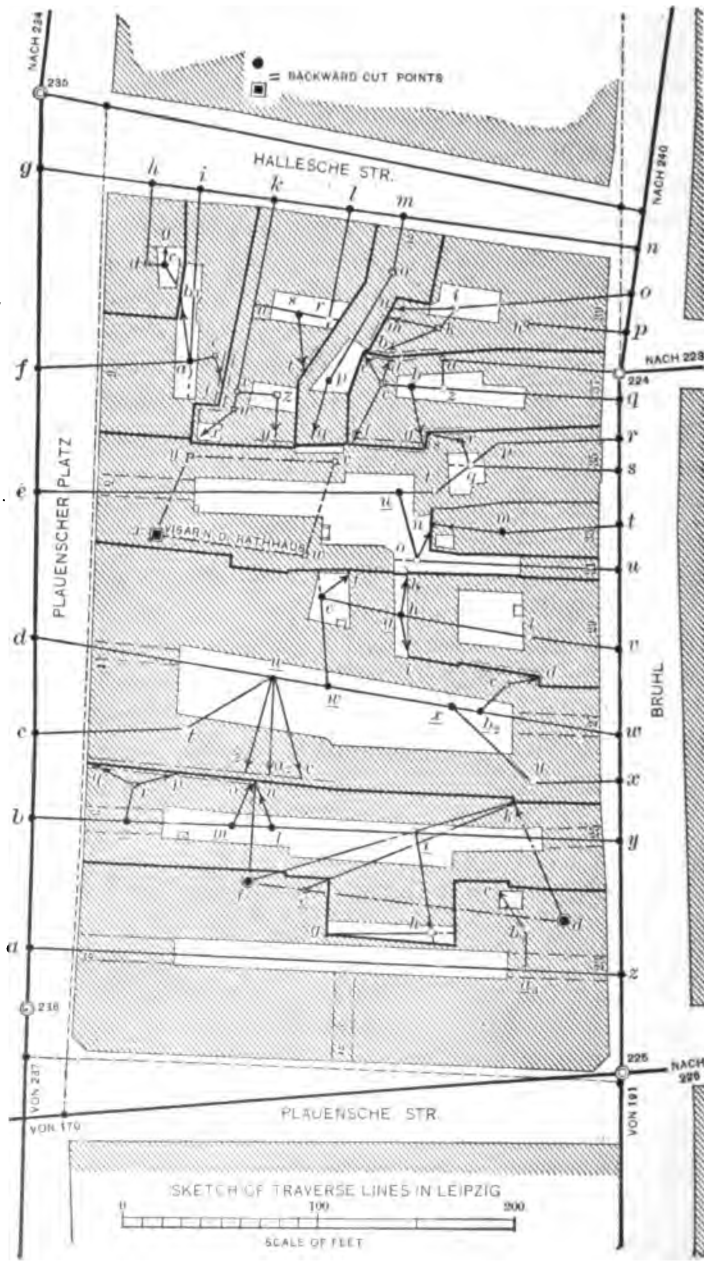
should not exceed 0·1 foot, and should in general be only one third of this amount. To ascertain the temperature at which a steel tape is to be regarded as of standard length, several lines trigonometrically determined can be measured in the same manner as ordinary traverse lines, and the temperature correction that will bring the closest harmony of results may afterward be used in deriving the correct standard temperature. In a similar investigation with wooden rods it has been found that rods regarded as 5 metres long should have an actual length of 5·0006 metres to obtain the best results.

Traverse lines of the third order should be doubly measured also, with maximum permissible differences of $0{\cdot}10\sqrt{d}$ between the two measurements, and, following the usual rule, mean differences of one third this amount. Where the two measurements of a line show too great a discrepancy they should both be rejected, and two new measurements should be made.

The accompanying sketch shows the network of traverse lines in one of the more simple blocks of the old portion of the city of Leipzig. Each block is separately treated. Before the commencement of work, notification is given to the owners or occupants by writing. The entire traverse work is generally laid out and the sides are measured before the beginning of the detail measurements; but in exceptional cases, to avoid repeated annoyances to the occupants, all the work is finished at one time. The points of the "block net" are all marked with stakes, oaken wedges driven into the joints of the pavement, or with nails or steel wires, the stakes and wedges being further marked with tacks. Upon the plan the wedges are indicated by a solid dot; stakes, by a circle with center point; nails, by small squares, etc.; thus rendering their recovery less difficult for the detail work. A free-hand sketch is drawn of the traverse lines, with lead pencil in the field, and afterward in the office with ink, with suitable red and blue lines. This sketch is, approximately, on a scale of from $\frac{1}{500}$ to $\frac{1}{1000}$. The net points follow each other in alphabetic series, and are designated a_o, b_o, c_o .... a_1, b_1, c_1, etc., the lettering beginning on the peripheral lines. Each block has its own lettering, beginning with a_o.

In the interior of the blocks, points merely interpolated on straight lines are distinguished from those at which the direction

FIG. 588.


= BACKWARD CUT POINTS
HALLESCHE STR.
PLAUENSCHER PLATZ
BRUHL
PLAUENSCHE STR.
NACH 224
NACH 240
NACH 223
NACH 226
VON 237
VON 170
VON 191
SKETCH OF TRAVERSE LINES IN LEIPZIG
0
100
200
SCALE OF FEET

changes by underscoring the letters. Angle measurements are made after the completion of the detail survey of the blocks by means of a party of four or five, the detail measurements employing only three.

885. Co-ordinate Computation. Printed blanks should be used for computing co-ordinates, five-place logarithms being employed, and the computation checked by the use of traverse tables. Gurden's or Defert's are the best,* the interpolations being effected with the slide rule.

The accompanying form shows the logarithmic computation of a traverse running from Point (253) to (259), both of which are previously located, and their co-ordinates are therefore given. The azimuths are known at starting: Rysedorph—(253) and at ending (259)—Capitol, S. E. By adding the angles, subtracting an even multiple of 180° from the sum, and comparing the result with the difference between the initial and final azimuths, the error of angle, 36″, is ascertained and distributed equally among the angles. The computation is then made with ink, the resulting error of co-ordinates is obtained and distributed as shown (corrected numbers being written in red ink, in such a manner as not to obscure the originals) before deriving the final co-ordinates. Co-ordinate corrections are made in one of two ways: first, as here shown, by distributing the errors in proportion to the magnitude of the individual co-ordinate differences, as compared with their absolute (not algebraical) sum; and, second, in proportion to the length of each traverse line as compared with the sum. The slide rule is very useful in effecting this distribution. The angle error, 36″ in the example, should not exceed $0{\cdot}3\sqrt{n}$ † minutes for traverse lines of the first and second order, and $0{\cdot}5\sqrt{n}$ for the tertiary traverse work, where n is the number of

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* Richard Lloyd Gurden, "Traverse Tables computed to Four Places of Decimals for every 1′ of Angle up to 100 of Distance," London, 1888. C. F. Defert, "Tafeln zur berechnung Rechtwinkliger Coordinaten," second edition, Berlin, 1874. Defert's tables are about equal to Gurden's in convenience, the typography is better, and they are less expensive. The five-place logarithmic tables [Gauss's] are more convenient than either.

† Although this is the form indicated by theory, it has been observed that with increasing n, the closure tends rather to diminish than to increase.

From Eagle Street, along north side of State Street, to Broadway. Traverse Lines.

Point.	Angle.	Azimuth, a	Distance, s	log. sin. a log. s log. cos. a	log. s sin. a log. s cos. a	+ s sin. a −	Y	+ s cos. a −	X	Point.
Ryse dorph.	° ′ ″	° ′ ″	Feet.				Feet.		Feet.	
		132 38 07								
(253)	351 51 29 6						+9671·11		−8903·99	(253)
		304 29 33	393·80	9·91603$_n$ 2·59528 9·75304	2·51181$_n$ 2·34832	324·57 4		223·01 3		
(254)	180 40 09 6						+9346·57		−8680·96	(254)
		305 09 39	198·70	9·91251$_n$ 2·29820 9·76033	2·21071$_n$ 2·05853	162·45 3		114·43 4		
(255)	180 26 09 6						+9184·14		−8566·52	(255)
		305 35 45	224·40	9·91016$_n$ 2·35102 9·76497	2·26118$_n$ 2·11599	182·47 5		130·62 3		
(256)	183 03 09 6						+9001·69		−8435·89	(256)
		308 38 51	93·00	9·89265$_n$ 1·96848 9·79556	1·86113$_n$ 1·76404	72·63 2		58·08		
(256½)							+8929·07		−8377·81	(256½)
		308 38 51	118·00	9·89265$_n$ 2·07188 9·79556	1·96453$_n$ 1·86744	92·16 5		73·79 1		
(257)	186 46 39 6						+8836·92		−8304·10	(257)
		315 25 27	430·84	9·84625$_n$ 2·63432 9·85267	2·48057$_n$ 2·48690	302·39 6		306·89 ·92		
(259)	350 30 39 6						+8534·56		−7997·18	(259)
	Error−36″	125 56 03	sum. 1458·34			Error−0·12		Error−0·08	$\sqrt{0\cdot12^2+0\cdot08^2}=$	0·145
Capitol S. E.									$\frac{0\cdot145}{1458\cdot34}=$	$\frac{1}{10000}$

angle points (6 in the example). The co-ordinate error of closure should in general never exceed $\frac{1}{2000}$ of the sum of the traverse sides, and should usually be within one third of this amount. The same formulas may also be used in ascertaining the greatest permissible linear discrepancy as were employed in deriving the greatest permissible difference between measured lines, namely, $0{\cdot}05\sqrt{d}$ for traverse lines of the first and second order, and $0{\cdot}10\sqrt{d}$, for traverse lines of the third order. The linear discrepancy is deduced from a comparison of the distance between the end points as previously known with that derived from the unadjusted traverse. After the final co-ordinates are ascertained the azimuths should again be computed, and should not differ by more than 24″ with traverse of the first and second order, and 40″ with traverse of the third order, from those originally used in the computation. After computing traverse lines of the first order, forming a closed polygon, containing generally several triangulation points on its periphery, it is sometimes well to carry the computations in all directions to a central "knot point" with the second order of traverse. The discordant values of the co-ordinates of the knot point may be averaged, and the errors then distributed in the usual manner among the radiating traverse lines.

886. Detailed Measurements. The detailed work rests on the traverse, and is to be recorded in the field on sketches made to an approximate and to a very large scale, to show details clearly and to give abundant room for figures. Sheets twelve by twenty inches, with details sketched on a scale of from 1 to 150 to 1 to 250, are to be recommended.

A stretched cord or a chalk line snapped on walk or pavement serves to measure abscissas, and ordinates are measured by means of a large wooden square or a right-angle prism.

In precise work the attempt is made to have all details located directly from the traverse lines, and not by measurements from one detail point to another.

887. Finished Plans. Finished maps may be made on a scale of 1 inch to 40 feet for ordinary city work, 1 inch to 20 feet for very

intricate portions, and 1 inch to 80 feet for general maps and for the suburbs; or the nearly equivalent fractional scales $\frac{1}{500}$, $\frac{1}{250}$, and $\frac{1}{1000}$ may be employed.* Whatever the scale, the sheets should be of the *same* size. They may be 0·8 metre × 0·6 metre, as in Berlin and Leipzig, or a little larger—1 metre × 0·66 metre. Three sheets of paper pasted together and well seasoned are sometimes used, or sheets of zinc may be used in place of the middle sheet of paper.

Such sheets will retain their size admirably.

888. Suburban Work. Suburban work is carried out by the usual topographical (tachymetrical) methods, with the use of stadia measurements and vertical angles for developing the contour lines. If a plane table is employed, considerable saving of time will be effected in field work on a large scale ($\frac{1}{1000}$) by doing as much preparatory work as possible with the aid of triangulation and traverse lines, and the stadia measurements should be confined to the location of contours, water courses, wooded areas, etc.

889. Lines of Levels. An excellent pattern of well-planned leveling is that of the city of Berlin. This was begun by leveling in a loop about the city. From this principal line ten main lines of levels were run, converging to a central point, with a greatest difference of 7·4 millimetres. The city was thus divided into polygons, which were again subdivided by other lines of levels. The lines were invariably leveled twice, with different observers, instruments, and rods.

The backsights and foresights were equal in length, and varied from 20 metres to 80 metres, averaging 50 metres.

The greatest permissible differences between the two lines of levels were: For distances of 50 metres, 2 millimetres; from 50 to 100 metres, 3 millimetres; from 500 to 750 metres, 6 millimetres; from 2,000 to 2,500 metres, 10 millimetres; from 11,000 to 12,000 metres, 20 millimetres.

The instruments and methods employed in refined geodetic work were used.

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* Suitable scales for this purpose, with the foot unit, may be obtained from the Brown & Sharpe Manufacturing Company of Providence, R. I. Maps thus made are easily adapted for use with the metric system.

Bench marks are best made with round-headed bolts fixed into walls of buildings with Portland cement. These should be set, located, and described before the leveling is begun.

890. Cost of Surveys. The methods above outlined for carrying on surveys have been well tested in many foreign cities. It may not be expedient with the surveys demanded in American cities to go into details with an equal degree of refinement, but it will be easy to diminish the degree of precision and to limit the scope of the survey where necessary. The survey of Berlin was begun in 1876, and, up to 1891, 19,718 ownerships with 39,819 buildings had been surveyed, with a vast amount of triangulation, leveling, computing, and other incidental work, at a cost of over $283,000. The city of Leipzig, with an area of about eleven square miles, appropriated $55,000 for a survey in 1884. After three years' progress, with all possible scientific refinement, it was estimated that the entire cost would be $76,000.

Similar work would doubtless cost double these amounts here. It would be unnecessary, on many accounts, to attempt the minute detail practiced in Germany, except in certain parts of our largest cities; but the preliminary work, triangulation, traverse lines, and levels will be necessary in any case. The city surveys undertaken in Germany may therefore be taken as patterns, and are worthy of careful study.

The methods and results of these surveys are almost invariably published with praiseworthy and painstaking care.

891. Indexing Records. Office records may be indexed best by means of one or more card indexes,* supplemented with index maps on a small scale, showing the location of different sheets of any given series of maps.

The card index has been developed for the use of libraries, and is the best and simplest means devised for enabling several indexes to be united without confusion by the aid of different colored cards. A liberal use of "guide cards" is essential in the card index. Important records should be duplicated, and so kept as to guard against

* The entire outfits for card indexes are offered in the most convenient styles by the Library Bureau of Boston and New York.

destruction by fire, etc. Where possible, maps should be kept flat, and in general a number of sheets should be preferred to a long roll. Sheets occupy less space than rolls, they can be more readily indexed and referred to, and they are much less liable to deterioration through age. The scale is also better preserved with flat sheets. Maps on stiff sheets of paper can conveniently be kept on edge in a vertical position in a suitable case. Great care should be taken to date all maps and records of every description, even the most trivial field notes; and the names of the draughtsmen or surveyors should be placed on the maps and record books made by each individual.

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THE END.